# § 10. Pullbacks & the wave equation

## § 10.1. Pullbacks by diffeomorphisms

Assume $U, V \subset \mathbb{R}^n$ are open.

**Defn.** A $C^1$ map $\Phi : U \to V$ is called a diffeomorphism if $\Phi$ is bijective and $\Phi^{-1} \in C^1$.

Similarly define $C^k$, $C^\infty$ diffeomorphisms.

**Jacobi's Formula:** if $\Phi : U \to V$ is a $C^1$ diffeomorphism and $f : V \to \mathbb{C}$ is measurable.

Then $f \in L^1(V) \Leftrightarrow f(\Phi(x))\,|\det d\Phi(x)| \in L^1(U)$ and in this case

$$\int_V f(y)\,dy = \int_U f(\Phi(x))\,|\det d\Phi(x)|\,dx$$

Think of this as a change of variables $y = \Phi(x)$

$|\det d\Phi(x)| > 0$ is called the Jacobian of $\Phi$ at $x$.

Assume now that
$\Phi: U \to V$ is a $C^\infty$ diffeomorphism.

We want to extend to distributions the pullback map
$$\Phi^*: L^1_{loc}(V) \to L^1_{loc}(U),$$
$$\Phi^* f := f \circ \Phi.$$

Take $f \in L^1_{loc}(V)$ and $\varphi \in C_c^\infty(U)$. Then
$$(\Phi^* f, \varphi) = \int_U f(\Phi(x))\,\varphi(x)\,dx$$
$$= \int_U \frac{f(\Phi(x))}{|\det d\Phi(x)|}\,\varphi(x)\,|\det d\Phi(x)|\,dx =$$
$$= \int_V \frac{f(y)}{|\det d\Phi(\Phi^{-1}(y))|}\,\varphi(\Phi^{-1}(y))\,dy$$
$$= (f, (\Phi^*)^t \varphi) \quad \text{where}$$
$$(\Phi^*)^t \varphi(y) = \frac{\varphi(\Phi^{-1}(y))}{|\det d\Phi(\Phi^{-1}(y))|}, \quad y \in V$$

Note that

$$(\Phi^*)^t : C_c^\infty(U) \to C_c^\infty(V).$$

**Defn** For $v \in \mathcal{D}'(V)$, define $\Phi^* v \in \mathcal{D}'(U)$ by

$$(\Phi^* v, \varphi) = (v, (\Phi^*)^t \varphi) \quad \forall \varphi \in C_c^\infty(U).$$

**Properties:**

1. $\Phi^* : \mathcal{D}'(V) \to \mathcal{D}'(U)$ is sequentially continuous
2. $\Phi^* f = f \circ \Phi$ for $f \in L^1_{loc}$
3. $\Phi^*$ is the unique operator satisfying 1 + 2
   since $C_c^\infty(V)$ is dense in $\mathcal{D}'(V)$
4. Chain Rule:

$$\partial_{x_j}(\Phi^* v) = \sum_{k=1}^n \Phi^*(\partial_{y_k} v) \cdot \partial_{x_j} \Phi_k$$

where $\Phi = (\Phi_1, \dots, \Phi_n)$.

**Proof** Immediate for $v \in C^\infty(V)$ (usual Chain Rule)
For general $v$, use that $C^\infty(V)$ is dense in $\mathcal{D}'(V)$. $\square$

## Examples:

1. $u \in \mathcal{D}'(\mathbb{R}^n)$ is homogeneous of degree $a$ $\iff$ $\Phi_t^* u = t^a u$ for all $t>0$ where $\Phi_t(x) = tx$, $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$.

2. Pullback of delta-function:

Say $\Phi : U \to V$ and $y_0 \in V$.

Then $\forall \varphi \in C_c^\infty(U)$ we have

$$(\Phi^*\delta_{y_0}, \varphi) = (\delta_{y_0}, (\Phi^*)^t \varphi) = (\Phi^{*t})\varphi(y_0)$$

$$= \frac{\varphi(\Phi^{-1}(y_0))}{|\det d\Phi(\Phi^{-1}(y_0))|}.$$

So $$\Phi^*\delta_{y_0} = \frac{\delta_{\Phi^{-1}(y_0)}}{|\det d\Phi(\Phi^{-1}(y_0))|}.$$

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## §10.2 Pullbacks by submersions

**Defn.** Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open where $n \geq m$. A <u>submersion</u> is a $C^\infty$ map $\Phi : U \to V$ such that $\forall x \in U$, $d\Phi(x) : \mathbb{R}^n \to \mathbb{R}^m$ is surjective.

**Thm** Assume that $\Phi : U \to V$ is a submersion. Then there exists a sequentially continuous operator $\Phi^* : \mathcal{D}'(V) \to \mathcal{D}'(U)$ such that $\Phi^* f = f \circ \Phi$ for all $f \in L^1_{loc}(V)$.

**Proof.** 1 Model case:

$\Phi_0(x', x'') = x'$ where we write elements of $\mathbb{R}^n$ as $(x', x'')$, $x' \in \mathbb{R}^m$, $x'' \in \mathbb{R}^{n-m}$.

Since $\Phi_0 : U \to V$, we have

$$U \subset V \times \mathbb{R}^{n-m}$$

For $f \in L^1_{loc}(V)$, we have

$$\Phi_0^* f(x', x'') = f(x') = f(x') \otimes 1(x''),$$

So for general $v \in \mathcal{D}'(V)$, define

$$\Phi_0^* v(x', x'') = v(x') \otimes 1(x'')\Big|_U$$

where $1 \in C^\infty(\mathbb{R}^{n-m})$.

Note by the way that $\forall \varphi \in C_c^\infty(U)$

$$(\Phi_0^* v, \varphi) = \int_{\mathbb{R}^{n-m}} (v(x'), \varphi(x', x''))\, dx''.$$

2 General case: any submersion locally looks like the model case.

More precisely, for each $x_0 \in U$ there exists open $U_{x_0} \subset U$, $x_0 \in U_{x_0}$ and a $C^\infty$ diffeomorphism

$$\varkappa_{x_0} : U_{x_0} \to W_{x_0} \subset \mathbb{R}^n$$ such that

on $U_{x_0}$, the map $\Phi$ is given by the first $m$ coordinates of $\varkappa_{x_0}$.

To see this, we apply the Inverse Mapping Thm:

- Since $d\Phi(x_0)$ is surjective, there exists a linear map $\Psi : \mathbb{R}^n \to \mathbb{R}^{n-m}$ such that the map

$$v \in \mathbb{R}^n \mapsto (d\Phi(x_0)v, \Psi(v)) \text{ is invertible.}$$

Put $\varkappa_{x_0}(x) := (\Phi(x), \Psi(x))$, then $d\varkappa_{x_0}(x_0)$ is invertible, so by the Inverse Mapping Thm $\varkappa_{x_0}$ is a diffeomorphism when restricted to some neighborhood of $x_0$.

We have

$$\Phi|_{U_{x_0}} = \Phi_0 \circ \varkappa_{x_0}.$$

Here $\varkappa_{x_0} : U_{x_0} \to W_{x_0}$ is a diffeomorphism

So $\varkappa_{x_0}^* : \mathcal{D}'(W_{x_0}) \to \mathcal{D}'(U_{x_0})$
was defined in §10.1

And $\Phi_0 : W_{x_0} \to V$, $\Phi_0(x', x'') = x'$;

So $\Phi_0^* : \mathcal{D}'(V) \to \mathcal{D}'(W_{x_0})$
was defined in Step 1 of the present proof.

So we can define

$$\Phi_{x_0}^* : \mathcal{D}'(V) \to \mathcal{D}'(U_{x_0})$$

sequentially continuous & such that
$\forall f \in L^1_{loc}(V)$,

$$\Phi_{x_0}^* f = (f \circ \Phi)|_{U_{x_0}}.$$

(3) For any $x_0, x_1$ and any $v \in \mathcal{D}'(V)$
we have $\Phi_{x_0}^* v|_{U_{x_0} \cap U_{x_1}} = \Phi_{x_1}^* v|_{U_{x_0} \cap U_{x_1}}$.

When $v \in C^\infty$ this is immediate since
both sides are just $(v \circ \Phi)|_{U_{x_0} \cap U_{x_1}}$.

For general $v$ this follows
since $C^\infty(V)$ is dense in $\mathcal{D}'(V)$.

Now from the sheaf property of distributions (see §2.3) we see that $\forall v \in \mathcal{D}'(V)$ there exists unique $\Phi^* v \in \mathcal{D}'(U)$ such that $\Phi^* v|_{U_{x_0}} = \Phi^*_{x_0} v \quad \forall\, x_0 \in U$.

This defines the operator $\Phi^*$ and it is direct to check that it has the needed properties. $\square$

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<u>Note:</u> the chain rule still holds for $\partial_{x_j}(\Phi^* v)$ when $\Phi: U \to V$ is a submersion (same proof)

<u>Example:</u> Assume that $\Phi: U \to \mathbb{R}$ is a submersion. If $H$ is the Heaviside function then $\Phi^* H = H \circ \Phi$ is the indicator function of the set $\Omega := \{x \in U \mid \Phi(x) > 0\}$

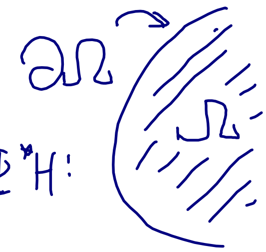
Then by the Chain Rule,

$$\partial_{x_j}(\Phi^* H) = \Phi^* H' \cdot \partial_{x_j} \Phi$$
$$= (\partial_{x_j} \Phi) \cdot \Phi^* \delta_0 .$$

What is $\Phi^* \delta_0$?

It will be supported on

$$\partial\Omega := \{x \in U \mid \Phi(x) = 0\}.$$




To compute it, we can actually use $\Phi^* H$:

$\forall \varphi \in C_c^\infty(U)$,

$$(\partial_{x_j}(\Phi^* H), \varphi) = -(\Phi^* H, \partial_{x_j}\varphi)$$

$$= -\int_\Omega \partial_{x_j}\varphi(x)\,dx =$$

(by Divergence Theorem)
$\partial_{x_j}\varphi = \operatorname{div}(\varphi e_j)$

$$= -\int_{\partial\Omega} \varphi(x)\,\nu_j(x)\,dS(x)$$

where $dS$ is the area measure on $\partial\Omega$ and $\vec{\nu} = (\nu_1, \dots, \nu_n)$ is the unit normal vector to $\partial\Omega$ pointing outside of $\Omega$

Denote by $\delta_{\partial\Omega} \in \mathcal{D}'(U)$ the distribution given by

$$(\delta_{\partial\Omega}, \varphi) = \int_{\partial\Omega} \varphi(x)\,dS(x)$$

We have $\vec{\nu} = -\dfrac{\nabla\Phi}{|\nabla\Phi|}$ because $\nabla\Phi$ is normal to $\partial\Omega$ & points <u>into</u> $\Omega$

So we get $\forall j$,

$$(\partial_{x_j}\Phi)\cdot\Phi^*\delta_0 = \partial_{x_j}(\Phi^* H) = $$
$$= \frac{\partial_{x_j}\Phi}{|\nabla\Phi|}\cdot\delta_{\partial\Omega}.$$

So (since $\forall x\in U\ \exists j: \partial_{x_j}\Phi\neq 0$)
we get
$$\Phi^*\delta_0 = \frac{1}{|\nabla\Phi|}\cdot\delta_{\partial\Omega}.$$

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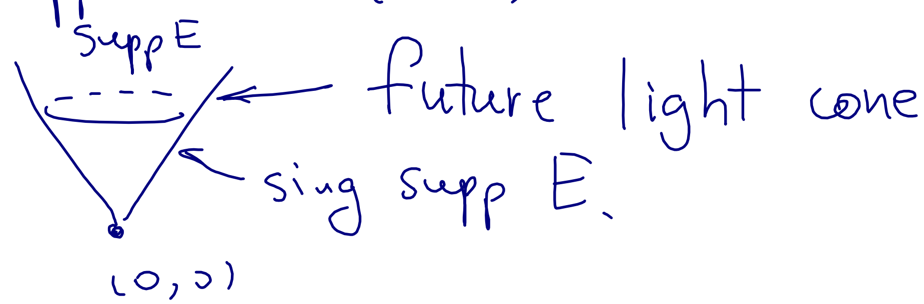
## §10.3. Wave equation in 3+1 dimensions

Wave operator in $\mathbb{R}^{n+1}$: $x_0 = \text{time}$, $x' = (x_1,\dots,x_n)$ space

$$\square = \partial^2_{x_0} - \partial^2_{x_1} - \dots - \partial^2_{x_n} = \partial^2_{x_0} - \Delta_{x'}$$

**Thm** $\square$ has a fundamental solution $E\in\mathcal{D}'(\mathbb{R}^{n+1})$ (called the advanced fundamental solution) such that:

1. $\operatorname{supp} E \subset \{(x_0,x') : |x'|\le x_0\}$
2. $\operatorname{sing\,supp} E = \{(x_0,x') : |x'| = x_0\}$



We already proved the Thm for $n=1$.

Now we will prove it for $n=3$ by constructing $E$.

See [Hörmander, § 6.2] for general $n$.

Consider the map

$$\Phi : \mathbb{R}^4 \to \mathbb{R}, \quad \Phi(x_0, x') = x_0^2 - |x'|^2.$$

It is a submersion on $\mathbb{R}^4 \setminus \{0\}$.

Take some $v \in \mathcal{D}'(\mathbb{R})$, $\Phi^* v \in \mathcal{D}'(\mathbb{R}^4 \setminus \{0\})$,

Compute $\Box(\Phi^* v)$ using the Chain Rule:

$$\partial_{x_0}(\Phi^* v) = 2x_0 \Phi^* v'$$

$$\partial_{x_j}(\Phi^* v) = -2x_j \Phi^* v' \qquad (j \geq 1)$$

$$\partial_{x_0}^2(\Phi^* v) = 2\Phi^* v' + 4x_0^2 \Phi^* v''$$

$$\partial_{x_j}^2(\Phi^* v) = -2\Phi^* v' + 4x_j^2 \Phi^* v''$$

So $\Box(\Phi^* v) = 8\Phi^* v' + 4\Phi \cdot \Phi^* v'' = \Phi^* w$

where $w(s) = 8v'(s) + 4s\, v''(s)$.

We want to take $v$ such that $w=0$, i.e. $s v''(s) + 2v'(s) = 0$.

If $v$ is homogeneous of degree $a$ then $v'$ is homogeneous of degree $a-1$, so by Euler's equation $s v''(s) = (a-1) v'(s)$

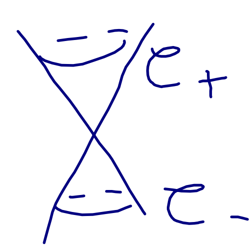
and $s v''(s) + 2 v'(s) = (a+1) v'(s)$

So we should take $v$ homogeneous of degree $-1$.

We will choose $v = \delta_0$, and we see that $\square (\Phi^* \delta_0) = 0$ in $\mathcal{D}'(\mathbb{R}^4 \setminus 0)$.

Now, $\operatorname{Supp}(\Phi^* \delta_0) \subset \{\Phi = 0\} = \{|x_0| = |x'|\}$
$= \mathcal{C}_+ \sqcup \mathcal{C}_-$ where $\mathcal{C}_\pm := \{x \in \mathbb{R}^4 \setminus \{0\} : x_0 = \pm |x'|\}$

And $\Phi^* \delta_0 = \dfrac{1}{|\nabla \Phi|} (\delta_{\mathcal{C}_+} + \delta_{\mathcal{C}_-})$




Now take just the $\mathcal{C}_+$ part:

$\tilde{E}_+ \in \mathcal{D}'(\mathbb{R}^4 \setminus \{0\})$,

$$\tilde{E}_+ := \frac{1}{|\nabla \Phi|} \delta_{\mathcal{C}_+}.$$

Parametrize $\mathcal{C}_+$ by $x' \in \mathbb{R}^3 \setminus \{0\}$:

$x_0 = |x'|$, $dS = \sqrt{1 + \left|\frac{\partial x_0}{\partial x'}\right|^2}\, dx' = \sqrt{2}\, dx'$

And $\nabla \Phi(x) = 2(x_0, -x')$,

$|\nabla \Phi(x)| = 2|x| = 2\sqrt{2}\,|x'|$ on $\mathcal{C}_+$, so

for all $\varphi \in C_c^\infty(\mathbb{R}^4 \setminus \{0\})$ we have

$$(\tilde{E}_+, \varphi) = \int_{\mathcal{C}_+} \frac{\sqrt{2}}{|\nabla \Phi_+|} \varphi(x)\, dS(x)$$

$$= \int_{\mathbb{R}^3 \setminus \{0\}} \frac{\varphi(|x'|, x')}{2|x'|}\, dx'.$$

Now we can extend $\tilde{E}_+$ to $\mathbb{R}^4$:

define $E_+ \in \mathcal{D}'(\mathbb{R}^4)$, $E_+\big|_{\mathbb{R}^4 \setminus \{0\}} = \tilde{E}_+$,

$\forall \varphi \in C_c^\infty(\mathbb{R}^4)$ we have

$$(E_+, \varphi) = \int_{\mathbb{R}^3} \frac{\varphi(|x'|, x')}{2|x'|}\, dx'$$

where the $\int$ converges.

We know:

- $\Box E_+ \big|_{\mathbb{R}^4\setminus\{0\}} = \Box \widetilde{E}_+ \big|_{\mathbb{R}^4\setminus\{0\}} = 0$

So $\operatorname{supp}(\Box E_+) \subset \{0\}$

which means $\Box E_+ = \sum_{|\alpha|\le N} c_\alpha \partial_x^\alpha \delta_0$

for some $N$, $c_\alpha \in \mathbb{C}$

- $E_+$ is homogeneous of degree $-2$:

if $\varphi_t(x) = t^4 \varphi(tx)$, $t>0$, then

$$(E_+, \varphi_t) = \int_{\mathbb{R}^3} \frac{t^4 \varphi(|tx'|, tx')}{2|x'|} dx' \qquad x' = \frac{y'}{t}$$

$$= t^2 \int_{\mathbb{R}^3} \frac{\varphi(|y'|, y')}{2y'} dy' = t^2 (E_+, \varphi).$$

- So $\Box E_+$ is homogeneous of degree $-4$.

Since $\partial^\alpha \delta_0$ is homogeneous of degree $-4-|\alpha|$ we see that

$$\Box E_+ = c\delta_0 \text{ for some } c \in \mathbb{C}.$$

We later compute $c \neq 0$, so $\boxed{E := c^{-1} E_+}$ is a fundamental solution of $\Box$.

• $\operatorname{sing\,supp} E_+ \subset \operatorname{supp} E_+ = \{x_0 = |x'|\}$

<u>So</u>: to prove the Thm it remains to compute $c$ in $\Box E_+ = c\delta_0$ & make sure that $c \neq 0$.

Take any $\psi \in C_c^\infty(-1,1)$,
$\chi \in C_c^\infty(\mathbb{R}^3)$, $\chi = 1$ near $\overline{B_{\mathbb{R}^3}(0,1)}$

and define $\varphi(x_0, x') := \psi(x_0)\chi(x') \in C_c^\infty(\mathbb{R}^4)$.

We have $(E_+, \Box\varphi) = (\Box E_+, \varphi) = c(\delta_0, \varphi)$
$= c\psi(0)$.

But $\Box\varphi = (\partial_{x_0}^2 - \Delta_{x'})\varphi$

$= \psi''(x_0)\chi(x') - \psi(x_0)\Delta\chi(x')$

Now $\operatorname{supp}(\psi(x_0)\Delta\chi(x')) \cap \operatorname{supp} E = \emptyset$

as this supp is in $|x_0| < 1$, $|x'| > 1$

So $(E_+, \Box\varphi) = (E_+, \psi''(x_0)\chi(x'))$

$$= \int_{\mathbb{R}^3} \frac{\psi''(|x'|)}{2|x'|}\,dx' = \text{(spherical coordinates)}$$

$$= 2\pi \int_0^\infty \psi''(r)\cdot r\,dr \overset{\text{IBP}}{=} 2\pi\psi(0).$$

So $c = 2\pi$ &
the fundamental solution $E$ to $\Box$
is $E = \frac{1}{2\pi} E_+$; $\forall \varphi \in C_c^\infty(\mathbb{R}^4)$,

$$(E, \varphi) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\varphi(|x'|, x')}{|x'|} \, dx' . \quad \Box$$

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We now give some basic corollaries
for the <u>Cauchy problem</u>

$$\begin{cases} \Box u(x_0, x') = f, & x_0 \geq 0 \\ u|_{x_0=0} = g_0(x) \\ \partial_{x_0} u|_{x_0=0} = g_1(x). \end{cases} \qquad (*)$$

We will only discuss <u>uniqueness</u>
and <u>singularities</u>. For more,
see [Hörmander, Thm. 6.2.4]

Assume that $u \in C^2(\{x_0 \geq 0\})$
$g_0 \in C^2(\mathbb{R}^n)$
$g_1 \in C^1(\mathbb{R}^n)$.

Solve $(*)$

Define $v(x_0, x') = H(x_0)\, u(x_0, x')$
$\uparrow$ Heaviside fn.

Then $v \in \mathcal{D}'(\mathbb{R}^{n+1})$, $\operatorname{supp} v \subset \{x_0 \geq 0\}$, and we compute (See Pset 5, Exercise 6)

$$\Box v = \delta_0'(x_0) \otimes g_0(x') + \delta_0(x_0) \otimes g_1(x') + H(x_0) f.$$

Now, let $E$ be the advanced fundamental solution of $\Box$.

Then $\operatorname{supp} E \subset \{|x'| \leq x_0\}$ and
$\operatorname{supp} v \subset \{x_0 \geq 0\}$
Sum properly (See Pset 5, Exercise 3)

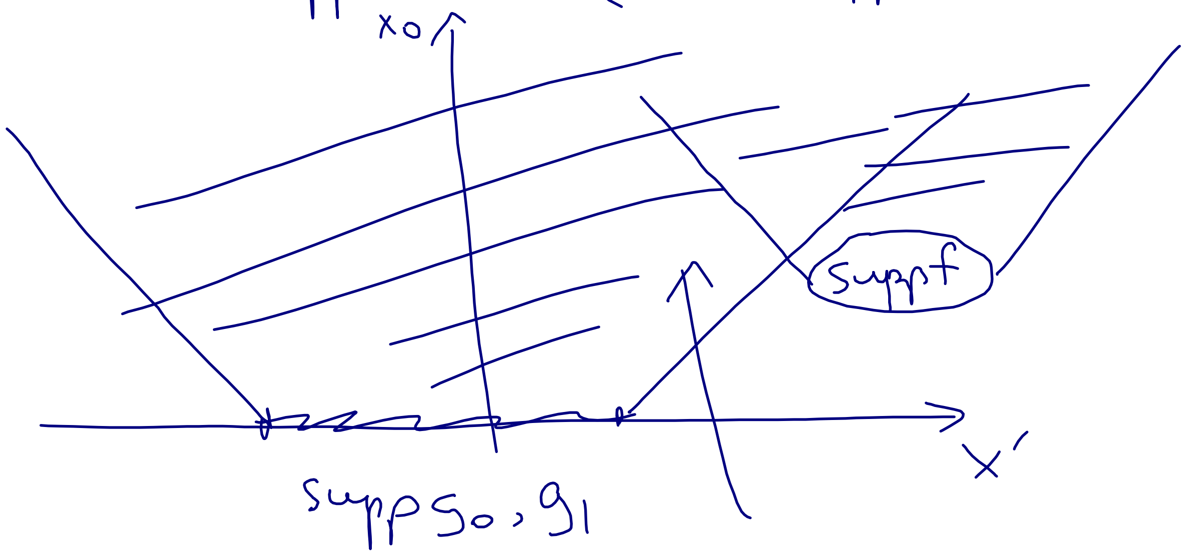
So $\boxed{v = E * \Box v}$

This gives <u>uniqueness</u> for the Cauchy problem (*): if $f = 0$, $g_0 = g_1 = 0$ then $\Box v = 0$, so $v = 0$ and thus $u = 0$.

We also get finite speed of propagation:

$\operatorname{supp} v \subset \operatorname{supp} E + \operatorname{supp}(\Box v)$

and $\operatorname{supp}(\Box v) = (0 \times (\operatorname{supp} g_0 \cup \operatorname{supp} g_1)) \cup \operatorname{supp} f$

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And propagation of singularities (weak form):

if $g_0 = g_1 = 0$ and $\operatorname{supp} f \subset \{x_1 > 0\}$

$\operatorname{sing\,supp} v \subset \operatorname{sing\,supp} E + \operatorname{sing\,supp} f$