18.118, SPRING 2022, PROBLEM SET 6

Review/useful information:

- Pushforward of a measure μ by a map φ: $\varphi_*\mu(A) = \mu(\varphi^{-1}(A))$.
- Schottky group: $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ generated by $\gamma_1, \dots, \gamma_m$ where, denoting $\mathcal{A} = \{1, \dots, 2m\}$ and $\bar{a} = a \pm m$,
$$\gamma_a(\dot{\mathbb{C}} \setminus D^\circ_{\bar{a}}) = D_a, \quad \gamma_{\bar{a}} = \gamma_a^{-1} \quad \text{for all} \quad a \in \mathcal{A}$$
where $D_1, \dots, D_{2m} \subset \mathbb{C}$ are disjoint closed disks centered on $\mathbb{R}$.
- Schottky words: for $\mathbf{a} = a_1 \dots a_n \in \mathcal{A}^n$, we say $\mathbf{a} \in \mathcal{W}^n$ if $a_{j+1} \neq \overline{a_j}$ for all j. In this case define $\gamma_{\mathbf{a}} := \gamma_{a_1} \circ \cdots \circ \gamma_{a_n} \in \Gamma$ and the disk $D_{\mathbf{a}} := \gamma_{a_1 \dots a_{n-1}}(D_{a_n})$.
- Limit set of a Schottky group: $\Lambda_\Gamma = \bigcap_{n \geq 1} \bigsqcup_{\mathbf{a} \in \mathcal{W}^n} D_{\mathbf{a}}$.
- You can use the following statement (it's not hard to prove but let's not bother): if $\mathrm{diam}(D_{\mathbf{a}})$ denotes the diameter of the disk $D_{\mathbf{a}}$ then
$$\max_{\mathbf{a} \in \mathcal{W}^n} \mathrm{diam}(D_{\mathbf{a}}) \to 0 \quad \text{as} \quad n \to \infty. \tag{1}$$
- Brouwer fixed point theorem: if $K \subset \mathbb{R}^n$ is a nonempty convex compact set and $B : K \to K$ is a continuous map, then there exists $x \in K$ such that $B(x) = x$.

1. Consider the map $\varphi(x) = 2x \bmod \mathbb{Z}$ on $X = \mathbb{R}/\mathbb{Z}$. In this exercise we see what happens in the construction of a measure of maximal entropy for a particular choice of separated points. Namely, take $n \geq 1$ and put
$$\nu_n := 2^{-n} \sum_{\ell=0}^{2^n - 1} \delta_{\ell/2^n}, \qquad \mu_n := \frac{1}{n} \sum_{j=0}^{n-1} \varphi^j_* \nu_n.$$
Find the weak limit of μ_n as $n \to \infty$.

2. Let Γ be a Schottky group. Define the set
$$\mathcal{D} := \mathbb{H}^2 \setminus \bigsqcup_{a \in \mathcal{A}} D^\circ_a.$$
Show that for each $z \in \mathbb{H}^2$, there exists $\gamma \in \Gamma$ such that $\gamma(z) \in \mathcal{D}$. (That is, every orbit of Γ on $\mathbb{H}^2$ intersects $\mathcal{D}$. This is part of what one needs to check to show that $\mathcal{D}$ is a fundamental domain of the action of Γ on $\mathbb{H}^2$. Hint: using (1), show that there exists n depending on z such that $z \notin D_{\mathbf{a}}$ for all $\mathbf{a} \in \mathcal{W}^n$.)

3. (Optional) Let Γ be a Schottky group and fix $z \in \mathbb{H}^2$. Let $x \in \mathbb{R}$. Show that x is the limit of a sequence of points in the orbit $\{\gamma(z) \mid \gamma \in \Gamma\}$ if and only if x lies in the limit set Λ_Γ defined above.

4. (Optional) Let $\mathcal{F}$ be the hyperbolic funnel $[0,\infty)_r \times \mathbb{S}^1_\theta$, where $\mathbb{S}^1 = \mathbb{R}/\ell\mathbb{Z}$ for some $\ell > 0$, with the metric $g = dr^2 + \cosh^2 r\, d\theta^2$. Assume that $\psi : [0,T] \to \mathcal{F}$ is a geodesic segment such that the endpoints $\psi(0)$ and $\psi(T)$ lie on the boundary circle $\{r = 0\}$. Show that the whole segment ψ lies on $\{r = 0\}$. (This implies that for a convex co-compact hyperbolic surface the convex core, which is the complement of all the interiors of the funnels, is geodesically convex. You can use any way you want to compute the geodesic flow on $\mathcal{F}$, including the contact flow approach of §5 in the lecture notes.)

5. In this exercise we prove a version of the Perron–Frobenius theorem for matrices. A generalization of this to transfer operators is given by the Ruelle–Perron–Frobenius Theorem (see e.g. Bowen, Theorem 1.7) which was not properly stated in class but parts of the proof were used to construct the Patterson–Sullivan measure.

Assume that A is a real $n \times n$ matrix such that all its entries a_{jk} are positive numbers. Denote by $\mathbf{1} \in \mathbb{R}^n$ the vector $(1,1,\dots,1)$ and by $\langle \bullet, \bullet \rangle$ the Euclidean inner product on $\mathbb{R}^n$. Define also the sets

$$\begin{aligned} \mathcal{C} &:= \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_j \geq 0 \quad \text{for all } j\}, \\ \mathcal{C}^\circ &:= \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_j > 0 \quad \text{for all } j\}. \end{aligned}$$

(a) Show that A has an eigenvector in $\mathcal{C}^\circ$: there exist $\lambda > 0$ and $v \in \mathcal{C}^\circ$ such that $Av = \lambda v$. Hint: first construct an eigenvector in the set $\{x \in \mathcal{C} \mid \langle x, \mathbf{1} \rangle = 1\}$, by applying the Brouwer fixed point theorem to the map $x \mapsto Ax/\langle Ax, \mathbf{1} \rangle$. Then show the following statement:

$$\text{If } x \in \mathcal{C} \setminus \{0\} \quad \text{is an eigenvector of } A \quad \text{then} \quad x \in \mathcal{C}^\circ. \tag{2}$$

(b) Show that there exists a constant C such that for all $\ell \geq 0$ we have $\|A^\ell\| \leq C\lambda^\ell$. (Hint: look instead at the norm of $(A^\ell)^*$ where A^* is the transpose of A. Using that $(A^\ell)^*$ has positive entries, show that this norm must be maximized by a vector in $\mathcal{C} \setminus \{0\}$. Then show that there exists a constant C such that for all $x \in \mathcal{C}$ we have $|x| \leq C\langle v, x \rangle$ and compute $\langle v, (A^\ell)^* x \rangle$.)

(c) Show that λ is a simple eigenvalue of A and every eigenvalue ν of A satisfies $|\nu| \leq \lambda$. (Hint: to exclude Jordan blocks and eigenvalues with $|\nu| > \lambda$, use part (b). To show that the eigenspace at λ is one-dimensional, argue by contradiction: if there are two linearly independent eigenvectors, then we can take their linear combination to produce an eigenvector which contradicts (2).)

On the next page is a plot of eigenvalues of a (somewhat) randomly chosen matrix with all positive entries. The leading eigenvalue λ is in red. Note that there are no eigenvalues other than λ on the circle of radius λ; this is true in general but we don't prove it in this exercise.