Review/useful information:

- Bowen metric for a map: if \((X,d)\) is a metric space and \(\varphi : X \to X\) is continuous, then
  \[
d_{n,\varphi}(x,y) = \max\{d(\varphi^j(x),\varphi^j(y)) \mid 0 \leq j < n\}.
  \]

- Topological entropy of a map: 
  \[
h_{\text{top}}(\varphi) = \lim_{\varepsilon \to 0^+} h_{\varepsilon}(\varphi)
  \]
  where
  \[
h_{\varepsilon}(\varphi) = \lim_{n \to \infty} \frac{\log D_{\varepsilon,n}(\varphi)}{n}
  \]
  and \(D_{\varepsilon,n}(\varphi)\) is the smallest number of sets which cover \(X\) and have \(d_{n,\varphi}\)-
diameter no more than \(\varepsilon\).

- Bowen metric for a flow: if \(\varphi^t : X \to X\) is a one-parameter continuous group of continuous maps then for \(T \geq 0\)
  \[
d_{T,\varphi}(x,y) = \sup\{d(\varphi^t(x),\varphi^t(y)) \mid 0 \leq t \leq T\}.
  \]

The topological entropy of a flow is defined similarly to that of a map.

- Entropy of a partition: if \(\xi = (A_\ell)_{\ell=1}^m\) and \(\mu\) is a probability measure then
  \[
  H_\mu(\xi) := -\sum_{\ell=1}^m \mu(A_\ell) \log \mu(A_\ell).
  \]

- Refined partition by a map \(\varphi\): if \(\xi = (A_\ell)_{\ell=1}^m\) then
  \[
  \xi^{(n)} := \bigvee_{j=0}^{n-1} \varphi^{-j}(\xi) = \left\{ \bigcap_{j=0}^{n-1} \varphi^{-j}(A_{w_j}) \mid w_0, \ldots, w_{n-1} \in \{1, \ldots, m\} \right\}.
  \]

- Entropy of a map \(\varphi\) with respect to a measure \(\mu\):
  \[
  h_\mu(\varphi) = \sup\{h_\mu(\varphi,\xi) \mid \xi \text{ a finite partition}\},
  \]
  \[
  h_\mu(\varphi,\xi) = \lim_{n \to \infty} \frac{H_\mu(\xi^{(n)})}{n}.
  \]

1. Let \(\varphi^t : X \to X\) be a continuous flow on a compact metric space \(X\). Show that the topological entropy of the flow \(\varphi\) is equal to the topological entropy of its time-one map \(\varphi^1\).

2. Let \(\varphi : X \to X\) be a diffeomorphism of a compact manifold \(X\). Show that \(h_{\text{top}}(\varphi)\) is finite. (Hint: you can bound it in terms of the Lipschitz constant of \(\varphi\) and the
dimension $m = \dim X$. You might want to use the fact that if $\mu$ is a smooth volume measure on $X$, then $\mu(B(x,r)) \geq C^{-1}r^m$ for all $x \in X$ and $0 < r < 1$.)

3. Let $\varphi : X \to X$ be an Anosov map (we assume that $\dim X > 0$ which implies that $\dim E_u, \dim E_s > 0$). Show that $h_{\text{top}}(\varphi) > 0$. You may use the following quantitative version of the Stable/Unstable Manifold Theorem: there exists $\lambda \in (0,1)$ such that for $\varepsilon > 0$ small enough and all $n \geq 0$, if $d_{\mu,\varphi}(x,y) \leq \varepsilon$ then $d(y,W^s(x)) \leq \lambda^n$, where $W^s(x)$ is the stable manifold centered at $x$. (It is actually possible to recover this statement from the version that we studied in class but you need not do this here.)

4. (Optional) Consider the map on $S^1 = \mathbb{R}/\mathbb{Z}$

$$\varphi : S^1 \to S^1, \quad \varphi(x) = 3x \mod \mathbb{Z}.$$ Let $X \subset [0,1]$ be the mid-third Cantor set. We think of it as a subset of $S^1$. Show that $\varphi(X) \subset X$ and compute the topological entropy of the restriction $\varphi|_X$.

5. Let $\varphi : X \to X$ be an Anosov diffeomorphism. As in Problem 3 of the previous problem set, denote by $Z_n$ the set of periodic points of $\varphi$ of period $n$. Show that for each $\delta > 0$

$$|Z_n| = \mathcal{O}(e^{(h_{\text{top}}(\varphi)+\delta)n}) \quad \text{as} \quad n \to \infty.$$ Assume that $\varphi : X \to X$ is a map preserving a probability measure $\mu$. Show that for each $k \geq 1$ we have $h_{\mu}(\varphi^k) = kh_{\mu}(\varphi)$ and, if $\varphi$ is invertible, then $h_{\mu}(\varphi^{-1}) = h_{\mu}(\varphi)$. (Hint: for the first part, show that $kh_{\mu}(\varphi,\xi) = h_{\mu}(\varphi^k,\tilde{\xi})$ for an appropriate choice of a partition $\tilde{\xi}$.)

6. (Optional) Let $X = \mathbb{R}/\mathbb{Z}$ and $\varphi(x) = 2x \mod \mathbb{Z}$. Find a sequence $\mu_k$ of $\varphi$-invariant probability measures on $X$ which converges weakly to some measure $\mu$ and $h_{\mu_k}(\varphi) = 0$ for all $k$, yet $h_{\mu}(\varphi) > 0$. This shows that entropy is not a continuous function on the space of measures with weak convergence. (Hint: try to take each $\mu_k$ to be supported on finitely many points, whose number grows with $k$.)

8. (Optional) Consider the map $\varphi$ and the set $X$ from Problem 4. Fix $0 < b < 1$. Let $\mu_b$ be the Bernoulli convolution which is a probability measure supported on $X$ with the following property: for each word $w = w_1 \ldots w_n$, where $w_1, \ldots, w_n \in \{0,2\}$, if

$$I_w := \left(\sum_{j=1}^{n} w_j 3^{-j}\right) + [0,3^{-n}]$$

is one of the intervals featured in the construction of the Cantor set $X$, then

$$\mu_b(I_w) = b^{k_w}(1-b)^{n-k_w} \quad \text{where} \quad k_w = \#\{j \in \{1,\ldots,n\} \mid w_j = 0\}.$$ (Such $\mu_b$ exists, is unique, and is $\varphi$-invariant, where the latter follows from the fact that $\varphi^{-1}(I_w) \cap X = I_{bw} \sqcup I_{2w}$. You do not need to check this in your solution. One
can think of $\mu_b$ as the distribution of the random variable $\sum_{j=1}^{\infty} \omega_j 3^{-j}$ where $\omega_j$ are i.i.d. Bernoulli random variables with $P(\omega_j = 0) = b$, $P(\omega_j = 2) = 1 - b$. Taking $b = \frac{1}{2}$ gives the standard Cantor measure on the mid-third Cantor set.) Compute $h_\mu(\varphi)$.

You may use what we did in lecture: $h_\mu(\varphi) = h_\mu(\varphi, \xi)$ if $\xi$ is a partition such that $\max\{\text{diam}(A) \mid A \in \xi^{(n)}\} \to 0$ as $n \to \infty$. 