18.118, SPRING 2022, PROBLEM SET 2

Review/useful information:

- Contact form on a $2d-1$-dimensional manifold $X$: a 1-form $\alpha$ on $X$ such that $\alpha \wedge (d\alpha)^{d-1} \neq 0$ everywhere.
- Reeb vector field for a contact 1-form: the unique $V \in C^\infty(X; TX)$ such that $\alpha(V) = 1$ and $\iota_V d\alpha = 0$.
- Anosov flow $\varphi^t = e^{tV} : X \to X$: $V$ has no fixed points, $X$ is compact, and for each $x \in X$ we have the flow/stable/unstable decomposition $T_x X = E_0(x) \oplus E_s(x) \oplus E_u(x)$ with $E_0(x) = \mathbb{R} V(x)$ and there exist $C, \theta > 0$ such that for all $(x,v) \in TX$

$$|d\varphi^t(x)v| \leq Ce^{-\theta|t|}|v|, \quad \begin{cases} t \geq 0, & v \in E_s(x), \\ t \leq 0, & v \in E_u(x). \end{cases}$$

1. Let $X$ be a manifold and $\varphi : X \to X$ be a diffeomorphism. Assume that $x_0 \in X$ is a periodic point for $\varphi$, that is $\varphi^r(x_0) = x_0$ for some $r \geq 1$. Show that the periodic orbit $\gamma = \{\varphi^j(x_0) \mid 0 \leq j < r\}$ is a hyperbolic set for $\varphi$ if and only if the linear homomorphism $d\varphi^r(x_0) : T_{x_0}X \to T_{x_0}X$ has no eigenvalues on the unit circle in $\mathbb{C}$.

2. Let $\tilde{X}$ be a compact manifold, $\tilde{\varphi} : \tilde{X} \to \tilde{X}$ be a diffeomorphism, and $\tilde{\mu}$ be a $\tilde{\varphi}$-invariant probability measure on $\tilde{X}$. Let $(X, \varphi^t)$ be the suspension of $\tilde{\varphi}$ with roof function $\tau \equiv 1$; here $X$ is glued from the cylinder $\tilde{X} \times [0,1]_{\tau}$. Show that $\varphi^t$ cannot be mixing with respect to the measure $\mu = \tilde{\mu} \times ds$, that is there exist $f, g \in L^2(X)$ such that

$$\int_X f(g \circ \varphi^t) \, d\mu \not\to \left(\int_X f \, d\mu\right) \left(\int_X g \, d\mu\right) \quad \text{as} \quad t \to \infty.$$ 

3. (Optional) This exercise shows that a suspension flow over a compact manifold is never a contact flow. Let $\tilde{X}$ be a compact $2d$-dimensional manifold and put $X := \tilde{X} \times (0,1)_{\tau}$. Show that the vector field $\partial_{\tau}$ cannot be the Reeb vector field of any contact 1-form on $X$. (Hint: assume that $\alpha$ is a contact form with Reeb vector field $\partial_{\tau}$. Show that $\alpha = ds + j^* \beta$ where $j : X \to \tilde{X}$ is the projection map and $\beta$ is a 1-form on $\tilde{X}$. Then show that such a contact form cannot be a contact form, by integrating $(d\beta)^{\wedge d}$ over $\tilde{X}$.)

4. Assume that $\varphi^t = e^{tV} : X \to X$ is an Anosov flow and the manifold $X$ is connected. Assume that $f \in C^1(X)$ is a $\varphi^t$-invariant function, that is $f \circ \varphi^t = f$ for all $t \in \mathbb{R}$. Show that $f$ is constant. (Hint: fix $(x,v) \in TX$. Using $\varphi^t$-invariance of $f$ for $t \geq 0$, show that $df(x)v = 0$ if $v \in E_s(x)$. Arguing similarly with $t \leq 0$, show that $df(x)v = 0$...
if \( v \in E_u(x) \). Show also that \( df(x)V(x) = 0 \), and conclude that \( f \) is constant. This is a baby version of Hopf’s argument that we will study soon.)

5. Assume that \( X \) is a compact manifold and \( \varphi^t = e^{tV} \) is an Anosov flow on \( X \), which is also a contact flow, i.e. \( V \) is the Reeb vector field of some contact 1-form \( \alpha \). Show that the kernel of \( \alpha \) is given by \( E_u \oplus E_s \). (Hint: use that \( \alpha \) is \( \varphi^t \)-invariant to show that \( \alpha(x)v = 0 \) for \( v \in E_u \) and for \( v \in E_s \).)