§9. Entropy

An entropy of a map/flow, roughly speaking, is a nonnegative number which describes how "complicated" the map/flow is, in the sense of the growth of the number of "different" trajectories of length $n$ as $n \to \infty$.

Very roughly speaking,

$$\# \text{(different trajectories)} \sim \exp(\text{entropy} \cdot n)$$

of length $\leq n$

§9.1. Topological entropy

Let us start with the case of maps (not necessarily invertible, for now):

- $(X,d)$ compact metric space
- $\psi : X \to X$ continuous map
Before we bring in $\Psi$, we discuss various ways of counting how many $\varepsilon$-sized balls pack in $X$ etc.

**Defn.** Let $(X,d)$ be a compact metric space. Let $\varepsilon > 0$. Define:

1. $S_d(\varepsilon) = \text{the minimal number } N \text{ of points } x_1, \ldots, x_N \subseteq X \text{ such that } X \subseteq \bigcup_{j=1}^N B_d(x_j, \varepsilon)$, where $B_d(a, \varepsilon) = \{ x \in X : d(x,a) \leq \varepsilon \}$ is a metric ball.

2. $D_d(\varepsilon) = \text{the minimal number } N \text{ of sets } B_1, \ldots, B_N \subseteq X \text{ such that } X \subseteq \bigcup_{j=1}^N B_j$ and $\text{diam } B_j \leq \varepsilon \quad \forall j$.

Here $\text{diam } B = \sup \{ d(x,y) : x, y \in B \}$.
\[ N_d(\varepsilon) = \text{the maximal number} \ N \text{ of points} \ x_1, \ldots, x_N \in X \text{ such that} \ d(x_j, x_k) \geq \varepsilon \ \forall j \neq k. \]

Each of these is a slightly different way of counting how many different points \( X \) has if we look at "resolution" \( \varepsilon \) (w.r.t. the metric \( d \)). They are all roughly equivalent:

**Lemma 1.** We have

1. \( D_d(2\varepsilon) \leq S_d(\varepsilon) \leq D_d(\varepsilon) \) and
2. \( N_d(2\varepsilon) \leq S_d(\varepsilon) \leq N_d(\varepsilon) \)

**Proof.** \( S_d(\varepsilon) \leq D_d(\varepsilon) \):

if \( X \subset \bigcup_{j=1}^{D_d(\varepsilon)} B_j \) and \( \text{diam } B_j \leq \varepsilon \), then take any \( x_j \in B_j \) and we have \( X \subset \bigcup_{j=1}^{D_d(\varepsilon)} B(x_j, \varepsilon) \) (as \( B_j \subset B(x_j, \varepsilon) \)).
$D_d(2\varepsilon) \leq S_d(\varepsilon)$:
if $X \subset \bigcup_{j=1}^{S_d(\varepsilon)} B(x_j, \varepsilon)$
then $X \subset \bigcup_{j=1}^{S_d(\varepsilon)} B_j$ where $B_j := B(x_j, \varepsilon)$
and diam$(B_j) \leq 2\varepsilon$.

$S_d(\varepsilon) \leq N_d(\varepsilon)$: if $x_1, \ldots, x_{N_d(\varepsilon)}$
are $\varepsilon$-separated (i.e. $d(x_j, x_k) > \varepsilon$ $\forall j \neq k$)
then $X \subset \bigcup_{j=1}^{N_d(\varepsilon)} B(x_j, \varepsilon)$.
Indeed, otherwise $\exists x \in X : d(x_j, x) < \varepsilon$ $\forall j$
but then $x, x_1, \ldots, x_{N_d(\varepsilon)}$ would be $\varepsilon$-separated
(contradicting the maximality of $N_d(\varepsilon)$)

$N_d(2\varepsilon) \leq S_d(\varepsilon)$: if $x_1, \ldots, x_{N_d(2\varepsilon)}$
are $2\varepsilon$-separated, then $\forall j \neq k,$
$x_j$ and $x_k$ cannot lie in the same ball
of radius $\varepsilon$. So we need at least $N_d(2\varepsilon)$
balls of radius $\varepsilon$ to cover
the points $x_1, \ldots, x_{N_d(2\varepsilon)}$. □
Remark: (won't be useful right away...) if $X$ is a (compact) manifold, \( \dim X = r \), then as $\varepsilon \to 0$ we have
\[
S_d(\varepsilon) \sim D_d(\varepsilon) \sim N_d(\varepsilon) \sim \varepsilon^{-r}
\]
in the sense that \( \forall C > 0 \forall \varepsilon < 1 \)
\[
\frac{1}{C} \varepsilon^{-r} \leq S_d(\varepsilon) \leq C \varepsilon^{-r} ...
\]

Now we bring in the map \( \varphi : X \to S \).
Fix a metric \( d \) on \( X \) and for \( n \geq 0 \)
define the refined metric
\[
d_n(x,y) = \max_{j=0}^{n-1} d(\varphi^j(x), \varphi^j(y))
\]
\[
= \max(d(x,y), d(\varphi(x), \varphi(y)), ..., d(\varphi^{n-1}(x), \varphi^{n-1}(y)))
\]
That is, \( d_n(x,y) \leq \varepsilon \) if
\[
\forall j = 0, ..., n-1, \ d(\varphi^j(x), \varphi^j(y)) \leq \varepsilon
\]
i.e. the trajectories \( \varphi^j(x), \varphi^j(y) \) stay \( \varepsilon \)-close for \( 0 \leq j < n \).
(If we are looking at a "resolution" ε then such trajectories look the same)

Define \( D_\varphi (\varepsilon, n) = D_{d_n} (\varepsilon) \)

= minimal number \( N \) of sets \( B_1, \ldots, B_N \) such that

\[ X \subset \bigcup_{l=1}^N B_l \quad \text{(they cover \( X \)) and} \]

\[ \forall x, y \in B_l, \ d_n (x, y) \leq \varepsilon \]

(each \( B_l \) has all trajectories ε-close for time \( 0 \leq j \leq n-1 \))

The quantity \( D_\varphi \) is useful to us because it is submultiplicative w.r.t. \( n \):

\textbf{Lemma 2} We have \( \forall n, m \geq 0 \)

\[ D_\varphi (\varepsilon, n+m) \leq D_\varphi (\varepsilon, n) D_\varphi (\varepsilon, m) \]

\textbf{Proof} Assume that \( D_{\varphi_l} (\varepsilon, n) \) \( l = 1 \), \( D_{\varphi_r} (\varepsilon, m) \) \( r = 1 \) are
collections of subsets such that
\[ X \subseteq \bigcup_{e \in E} B_e, \quad X \subseteq \bigcup_{r \leq r} C_r, \]
and \( \text{diam}_d(B_e) \leq \varepsilon \quad \forall e, r \)
\( \text{diam}_w(C_r) \leq \varepsilon \)

Define the sets \( (A(e, r)) \) as \( A(e, r) := B_e \cap \varphi^{-n}(C_r) \).
That is, \( x \in A(e, r) \) if \( x \in B_e \) and \( \varphi^n(x) \in C_r \).

Roughly speaking, \( e \) encodes what happens to the trajectory \( \varphi^k(x) \) for \( 0 \leq k < n \)
and \( r \) encodes what happens for \( n \leq k < n + m \).

We have \( X \subseteq \bigcup_{e, r} A(e, r) \), so we are done once we prove that
\( \forall x, y \in A(e, r), \ d_{n+m}(x, y) \leq \varepsilon. \)
But here
\[ d_{n+m}(x,y) = \max(d(x,y), \ldots, d(y^{n+m-1}(x), y^{n+m-1}(y))) = \max(d_n(x,y), d_m(\varphi^n(x), \varphi^n(y))) \]
and \( d_n(x,y) \leq \varepsilon \) (as \( x, y \in \mathcal{B} \mathcal{E} \))
and \( d_m(\varphi^n(x), \varphi^n(y)) \leq \varepsilon \) (as \( \varphi^n(x), \varphi^n(y) \in \mathcal{C} \mathcal{R} \)).

\[ \square \]

We now want to describe the asymptotics of \( D_\varepsilon(\varepsilon, n) \) as \( n \to \infty \).
I.e. how many "different at resolution \( \varepsilon \)" trajectories of length \( n \) are there?

We are hoping for something like
\[ D_\varepsilon(\varepsilon, n) \sim \varepsilon^n \text{ for some } \varepsilon \geq 0. \]

So we should define
\[ h_\varepsilon := \lim_{n \to \infty} \frac{\log D_\varepsilon(\varepsilon, n)}{n}. \]
Here the limit exists since \( \log D_p(\varepsilon, n) \) is subadditive (by Lemma 2) and by Fekete's Lemma. Assume an EIR is a subadditive sequence, i.e.,
\[ a_{n+m} \leq a_n + a_m \quad \forall n, m \geq 1. \]

Then
\[ \lim_{n \to \infty} \frac{a_n}{n} = \inf \frac{a_n}{n}. \]

Proof: Certainly
\[ \lim \inf_{n \to \infty} \frac{a_n}{n} \geq \inf \frac{a_n}{n}. \]

So it suffices to show that
\[ \lim \sup_{n \to \infty} \frac{a_n}{n} \leq \inf \frac{a_m}{m}. \]

For that it is enough to show that \( \forall m, \)
\[ \lim \sup_{n \to \infty} \frac{a_n}{n} \leq \frac{a_m}{m}. \]

Take large \( n \) & write \( n = q \cdot m + r, \quad 0 \leq r < m \)

Then \( a_n \leq q \cdot a_m + a_r. \)
So \[ \frac{a_n}{n} \leq \frac{q \cdot a_m + a r}{n} \leq \frac{q m}{n} \cdot \frac{a_m}{m} + \frac{a r}{n} \].

But as \( n \to \infty \), we have \( \frac{a r}{n} \to 0 \)

(only finitely many options for \( r \))

and \( \frac{q m}{n} \to 1 \). So

\[ \limsup_{n \to \infty} \frac{a_n}{n} \leq \frac{a_m}{m} \] as needed. \( \square \)

So now we have defined

\[ h_\varepsilon := \lim_{n \to \infty} \frac{1}{n} \log D_\varphi(\varepsilon, n). \]

Of course this can depend on \( \varepsilon \),

e.g. if \( \text{diam}(X) \leq \varepsilon \) then \( h_\varepsilon = 0 \) (as \( D_\varphi(\varepsilon, h) = 1 \))
Note that $h_\varepsilon$ is a decreasing function of $\varepsilon > 0$:

If $\varepsilon_1 < \varepsilon_2$ then $\forall n$

$D_\varphi(\varepsilon_1, n) \geq D_\varphi(\varepsilon_2, n)$

and thus $h_{\varepsilon_1} \geq h_{\varepsilon_2}$.

**Defn** The topological entropy of $\varphi$ is defined as

$h_{top}(\varphi) = \lim_{\varepsilon \to 0^+} h_\varepsilon = \sup_{\varepsilon > 0} h_\varepsilon,$

i.e. $h_{top}(\varphi) = \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \frac{1}{n} \log D_\varphi(\varepsilon, n)$.

Note: $h_{top}(\varphi)$ does not change if we replace $d$ by an equivalent metric (exercise)
Lemma 3 ① If \( \psi^k \), \( k \geq 1 \), is \( k \)-th iterate of \( \psi \), then
\[
h_{\text{top}}(\psi^k) = k \cdot h_{\text{top}}(\psi)
\]
② If \( \psi \) is a homeomorphism (i.e. \( \psi^{-1} \) exists & is continuous) then
\[
h_{\text{top}}(\psi^{-1}) = h_{\text{top}}(\psi)
\]

Proof ① Assume that \( BCX \)
is such that \( \text{diam}_{n_k, \psi}(B) \leq \varepsilon \), i.e. \( \forall j = 0, 1, \ldots, n_k - 1 \), we have \( \forall x, y \in B \)
\[
d(\psi^j(x), \psi^j(y)) \leq \varepsilon.
\]
Then \( \forall k = 0, \ldots, n - 1 \) we have \( \forall x, y \in B \)
\[
d(\psi^k(x), \psi^k(y)) \leq \varepsilon \] (putting \( j = k \varepsilon \)).
So \( \text{diam}_{n, \psi^k}(B) \leq \varepsilon \).
This shows \( D_{\psi^k}(\varepsilon, n) \leq D_{\psi}(\varepsilon, n_k) \).
Thus \( \frac{1}{n} \log D_{\phi^k}(\varepsilon, \eta) \leq \frac{k}{nk} \log D_{\phi}(\varepsilon, \eta) \tag{9-13} \)

Which (taking \( n \to \infty \)) gives

\[
\varepsilon (\phi^k) \leq k \varepsilon (\phi)
\]

and thus (taking \( \varepsilon \to 0 \))

\[
\varepsilon f_{\phi^k} (\psi) \leq k \varepsilon f_{\phi^k} (\psi).
\]

On the other hand, if we fix \( \varepsilon > 0 \) then there exists \( \tilde{\varepsilon} > 0 \) such that \( \forall x, y \in X, \)

\[
d(x, y) \leq \tilde{\varepsilon} \implies d(\psi^r(x), \psi^r(y)) \leq \varepsilon \quad \forall r = 0, \ldots, k - 1
\]

(by uniform continuity of \( \phi \)).

If \( BCX \) has \( \text{diam}_{\varepsilon_n, \phi^k} B \leq \varepsilon \),

i.e. \( \forall x, y \in B \) \( \forall r = 0, \ldots, n - 1 \)

\[
d(\psi^{k^r}(x), \psi^{k^r}(y)) \leq \varepsilon,
\]
Then $\forall x, y \in B$ $\forall j = 0, 1, \ldots, nk - 1$ we get $d(\varphi_j(x), \varphi_j(y)) \leq \varepsilon$ by writing $j = kl + r$, $0 \leq r < k$ and using that $d(\varphi^{kl}(x), \varphi^{kl}(y)) \leq \tilde{\varepsilon}$. It follows that

$$\text{diam}_{\text{nk}, \varphi} (B) \leq \varepsilon.$$ 

Thus $D_\varphi (\varepsilon, nk) \leq D_{\varphi^k} (\tilde{\varepsilon}, n)$ and (taking $\frac{1}{n} \log (\ldots)$ & $n \to \infty$)

$$k \cdot h_\varepsilon (\varphi) \leq h_{\tilde{\varepsilon}} (\varphi^k)$$

which gives (taking sup over $\varepsilon > 0$)

$$k \cdot h_{\text{top}} (\varphi) \leq h_{\text{top}} (\varphi^k).$$
Assume now that $\varphi$ is a homeomorphism. We show that
\[ h_{\text{top}}(\varphi^{-1}) \leq h_{\text{top}}(\varphi) \]
which is enough since we can apply the same argument to $\varphi^{-1}$.

It is enough to prove that $\forall \varepsilon > 0$, $n$ we have
\[ D_{\varphi^{-1}}(\varepsilon, n) \leq D_{\varphi}(\varepsilon, n). \]
Let $N := D_{\varphi}(\varepsilon, n)$ and $B_1, \ldots, B_n \subset X$ be such that $X \subset \bigcup_{e} e$ and $\operatorname{diam}_{n, \varphi^{-1}}(B_e) \leq \varepsilon$.

Put $\tilde{B}_e := \varphi^{n-1}(B_e)$. Then $X \subset \bigcup_{e} \tilde{B}_e$ and $\operatorname{diam}_{n, \varphi^{-1}}(\tilde{B}_e) \leq \varepsilon$.

Indeed, if $\tilde{x}, \tilde{y} \in \tilde{B}_e$, then
\[ \tilde{x} = \varphi^{n-1}(x), \quad \tilde{y} = \varphi^{n-1}(y) \quad \text{for some} \quad x, y \in B_e \]
and $\operatorname{diam}_{n, \varphi}(B_e) \leq \varepsilon$ implies $d(\varphi^j(x), \varphi^j(y)) \leq \varepsilon$ $\forall j = 0, \ldots, n-1$.

As $\varphi^{-k}(\tilde{x}) = \varphi^{n-1-k}(x) \Rightarrow d_{n, \varphi^{-1}}(\tilde{x}, \tilde{y}) \leq \varepsilon$. 
It follows that $D_{\varphi}^{-1}(\varepsilon, n) \leq N$ and finishes the proof. □

For flows, entropy is defined similarly to maps: if $\varphi^t : \mathcal{X} \to \mathcal{X}$ is a flow then we consider the metrics $d_T(x, y) = \sup_{0 \leq t \leq T} d(\varphi^t(x), \varphi^t(y))$, and we can define $D_{\varphi}(\varepsilon, T)$ and $h_{top}(\varphi)$.

§9.2. Examples of entropy computation

1. $X = \mathbb{S}^2 = \mathbb{R}/\mathbb{Z}$, $\varphi(x) = x + r \mod \mathbb{Z}$ for some fixed $r \in \mathbb{R}$:
   we have $d_n(x, y) = \max_{j=0}^{n-1} d(\varphi^j(x), \varphi^j(y))$,
   $= d(x, y)$ (where $d$ is the standard metric on $\mathbb{S}^2$)

Since $d(\varphi^j(x), \varphi^j(y)) = d(x, y) \forall j$, thus $D_{\varphi}(\varepsilon, n)$ is independent of $n$, so $h_{\varphi} = 0 \forall \varepsilon > 0$, and thus $h_{top}(\varphi) = 0$.

This works any time $\varphi$ is an isometry.
2. \( X = S^1 \), \( \varphi(x) = 2x \mod \mathbb{Z} \).

We use the following

**Lemma**  If \( \varepsilon \leq \frac{1}{4} \) and \( x, y \in S^1 \)
then \( d_n(x, y) \leq \varepsilon \iff d(x, y) \leq \frac{\varepsilon}{2^{n-1}} \).

**Proof**  We need to show
\[
d(x, y) \leq \frac{\varepsilon}{2^{n-1}} \iff \forall j = 0, \ldots, n-1, \ d(\varphi^j(x), \varphi^j(y)) \leq \varepsilon.
\]

\( \Rightarrow \): follows from the Lipschitz bound
\[
d(\varphi(x), \varphi(y)) \leq 2 \cdot d(x, y) \quad \forall x, y \in S^1.
\]

\( \Leftarrow \): we have \( \forall x, y \in S^1 \):

- If \( d(x, y) \leq \frac{1}{4} \) then
  \[
d(\varphi(x), \varphi(y)) = 2 \cdot d(x, y)
  \]

Thus, if \( d_n(x, y) \leq \varepsilon \leq \frac{1}{4} \) then
\[
d(\varphi^j(x), \varphi^j(y)) = 2^j \cdot d(x, y) \quad \forall j = 0, \ldots, n-1
\]
and thus \( d(x, y) \leq \frac{\varepsilon}{2^{n-1}} \).
Given the lemma, we see that \\
\forall \varepsilon \leq \frac{1}{n}, \ \forall n, \ we \ have \\
D(\varepsilon, n) = \text{minimal number } N \\
of \text{sets } B_1, \ldots, B_N \ s.t. \ S^1 \subset \cup B_e \ \text{and } \ \text{diam}_{d}(B_e) \leq \frac{\varepsilon}{2^{n-1}}. \\
\text{diameter w.r.t. the usual metric on } \mathbb{R}/2. \\
This \ gives \ D(\varepsilon, n) = \left\lceil \frac{2^{n-1}}{\varepsilon} \right\rceil \ \\
Then \ h_{\varepsilon} = \lim_{n \to \infty} \log \frac{D(\varepsilon, n)}{n} = \log 2 \\
So \ h_{\text{top}}(\varphi) = \log 2 \\

(3) Hyperbolic toral automorphisms: \\
X = \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2, \ \varphi(x) = Ax \mod \mathbb{Z}^2, \ \ A \in \text{SL}(2, \mathbb{Z}) \ \text{hyperbolic}. \\
Let \ \lambda, \lambda^{-1} \ \text{be the eigenvalues of } A, \ \text{with } |\lambda| > 1.
Fix some \( x \in X \) and \( n \geq 0 \), as well as small enough \( \varepsilon > 0 \).

We want to understand the ball
\[
\left\{ y \in X \mid d_n(x, y) \leq \varepsilon \right\}.
\]

Certainly, it's contained in the small ball \( B_d(x, \varepsilon) \), so we can locally identify \( \mathbb{T}^2 \) with \( \mathbb{R}^2 \) here.

Then we can define the stable/unstable distances \( d_s(x, y) \), \( d_u(x, y) \) as follows:

if \( v_u, v_s \in \mathbb{R}^2 \) are unit length vectors s.t. \( Av_u = \lambda v_u \), \( Av_s = \lambda^{-1} v_s \),

then write \( x-y = a_u v_u + a_s v_s \)

for some small \( a_u, a_s \in \mathbb{E} \mathbb{R} \) and put

\[
d_u(x, y) := |a_u| \\
d_s(x, y) := |a_s|.
\]
Similarly to the Lemma in Example 2, we have

**Lemma** If \( \varepsilon \) is small enough, then there exists \( C \) (independent of \( \varepsilon \)) such that

\[
\forall x, y \in X, \forall n \geq 0 \text{ we have }
\]

\( d_n(x, y) \leq \frac{\varepsilon}{C \cdot |K|^n}, \quad d_s(x, y) \leq \frac{\varepsilon}{C} \)

\( d_n(x, y) \leq \varepsilon \)

and

\( d_n(x, y) \leq \varepsilon \implies \)

\( \Rightarrow d_u(x, y) \leq \frac{C_{\varepsilon}}{|K|^n}, \quad d_s(x, y) \leq C_{\varepsilon}. \)

**Proof:** (a) Let's look at the set of \( y \in X: d_u(x, y) \leq \frac{\varepsilon}{C \cdot |K|^n}, \quad d_s(x, y) \leq \frac{\varepsilon}{C}. \) (\( \star \))

Looks like a rectangle.
Apply \( \Phi \) to this rectangle → another rectangle

\[ \frac{\varepsilon}{CN} \leq \frac{\varepsilon}{CN^{n-1}} \]

and so on up until \( \Phi^{n-1} \) (this set) is yet another rectangle.

(Note: no wrapping around the torus as \( \varepsilon \) is small)

Each of these sets has diameter \( \leq \varepsilon \) if \( C \) is large enough.

So if \( y \) satisfies (*) then

\[ d_n(x,y) \leq \varepsilon. \]
6) Assume now that $d_n(x,y) \leq \varepsilon$.

Then we can see (similarly to the pictures in part 2), and using that $\varepsilon$ is small) that for $j=0, \ldots, n-2$

$d_u(\varphi^{j+1}(x), \varphi^{j+1}(y)) = |N| \cdot d_u(\varphi^j(x), \varphi^j(y))$

$d_s(\varphi^{j+1}(x), \varphi^{j+1}(y)) = |N|^{-1} \cdot d_s(\varphi^j(x), \varphi^j(y))$

Indeed, write $x-y$ as a linear combination of $V_u, V_s$ and apply the linear map $A$. (No wrapping around since $\varepsilon$ is small).

So then: $d_u(\varphi^{n-1}(x), \varphi^{n-1}(y)) = |N|^{-1} \cdot d_u(x,y)$

And $d_u(\varphi^{n-1}(x), \varphi^{n-1}(y)) \leq C d(\varphi^{n-1}(x), \varphi^{n-1}(y)) \leq C \varepsilon$.

So $d_u(x,y) \leq \frac{C \varepsilon}{|N|}$. And $d_s(x,y) \leq C d(x,y) \leq C \varepsilon$. □
Now, \( D_{\mathcal{E}}(\varepsilon, n) \) is the minimal number \( N \) of sets \( B_1, \ldots, B_N \) s.t. \( X \subset \bigcup_{i \in \mathcal{E}} B_i \) and \( x, y \in B_i \Rightarrow d_n(x, y) \leq \varepsilon \).

* Each such \( B_i \) is contained in a rectangle (for any \( x \in B_i \))
  \[ \{ y : d_n(x, y) \leq \frac{C\varepsilon}{N^n}, d_s(x, y) \leq C\varepsilon \} \]
which has area \( \leq \frac{C'}{N^n} \).

So \( N \geq \frac{N^n}{C' \cdot \varepsilon^2} \).

* On the other hand, if \( C' \) is large enough then we can find \( N \leq \frac{C' N^n}{\varepsilon^2} \)
rectangles \( B_1, \ldots, B_N, \ X \subset \bigcup_{i \in \mathcal{E}} B_i \)
such that \( \text{diam}_{du}(B_i) \leq \frac{\varepsilon}{CN^n}, \ \text{diam}_{ds}(B_i) \leq \frac{\varepsilon}{C} \)
\[ \frac{\varepsilon}{CN^n} \text{ stable} \]
\[ \frac{\varepsilon}{C} \text{ unstable} \]
and thus \( \text{diam}_{d_n}(B_i) \leq \varepsilon \).
So, \( \frac{A^n}{C' \varepsilon^2} \leq D_n(\varepsilon, n) \leq \frac{C' A^n}{\varepsilon^2} \).

Then \( h_\varepsilon = \lim_{n \to \infty} \frac{\log D_n(\varepsilon, n)}{n} = \log |\chi| \).

Thus \( h_{\text{top}}(\varphi) = \log |\chi| \).

4. Geodesic flow \( \varphi^t : X \to X \)
where \( X = \text{SM} \) and \( (M, g) \)
is a hyperbolic surface.

A similar argument to (3) shows that \( h_{\text{top}}(\varphi) = 1 \)

Where we use that the expansion/contraction rate of the flow on the unstable/stable spaces is equal to 1: \( d\varphi^t \cdot U_\pm = e^{\pm t} U_\pm \).
We finish this subsection with the following Fact: if \( \varphi: X \to X \) is an Anosov map and
\[
N(T) = \# \text{ of periodic points of } \varphi \text{ of period } \leq T,
\]
then \( N(T) \sim e^{h_{\text{top}}(\varphi) \cdot T} \),
i.e. \[
\lim_{T \to \infty} \frac{1}{T} \log N(T) = h_{\text{top}}(\varphi).
\]
A similar statement holds for Anosov flows.

We won't give a proof; see [Katok–Hasselblatt, §18.5].