§14. Dynamical zeta function

We continue working with a <u>convex co-compact hyperbolic surface</u> $M = \Gamma \backslash \mathbb{H}^2$ where $\Gamma \subset PSL(2, \mathbb{R})$ is a <u>Schottky group</u>.

§14.1. Selberg zeta function

Let $\varphi^t : SM \circlearrowleft$ be the geodesic flow.

A <u>closed geodesic</u> on M corresponds to a pair $((x,v), T)$ where $(x,v) \in SM$, $T > 0$, and

$$\varphi^T(x,v) = (x,v).$$

We identify $((x,v), T)$ with

$$(\varphi^t(x,v), T) \quad \forall\, t \in \mathbb{R}.$$

Note that we necessarily have

$$(x,v) \in K \leftarrow \text{trapped set}.$$

We say $((x,v),T)$ is a <u>primitive</u> closed geodesic, if

$$\varphi^t(x,v) \neq (x,v) \text{ for all } t \in (0,T).$$

One can show that $\exists C \, \forall R$ the number of closed geodesics of period $\leq R$ is $O(e^{CR})$.
(This can be proved using the Stable/Unstable manifold Thm for the geodesic flow on K similarly to Pset 4, Exercise 3).

<u>Defn.</u> For $s \in \mathbb{C}$, $\operatorname{Re} s > C$, define the <u>Selberg zeta function</u>

$$Z_M(s) = \prod_T \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)T}\right)$$

where $\prod_T$ is over the periods of primitive closed geodesics, with multiplicity.

For $\mathrm{Re}\, s > c$, the product

converges since

$$\sum_T \sum_{k=0}^{\infty} \left| e^{-(s+k)T} \right| < \infty :$$

Split into pieces (assume for simplicity $T \geq 1$ always)

$$r \leq T < r+1 , \quad r \in \mathbb{N},$$

at most $O(e^{Cr})$ terms T

in each piece, so get a bound by

$$\sum_{r=1}^{\infty} \sum_{k=0}^{\infty} e^{Cr} \left| e^{-(s+k)r} \right|$$

$$= \sum_{r=1}^{\infty} \sum_{k=0}^{\infty} e^{(C - \mathrm{Re}\, s - k)r} =$$

(geometric progression) $C - \mathrm{Re}\, s - k < 0$

$$= \sum_{k=0}^{\infty} \frac{e^{C - \mathrm{Re}\, s - k}}{1 - e^{C - \mathrm{Re}\, s - k}} < \infty .$$

Note: $Z_M(s)$ is holomorphic in s when $\mathrm{Re}\, s > C$.

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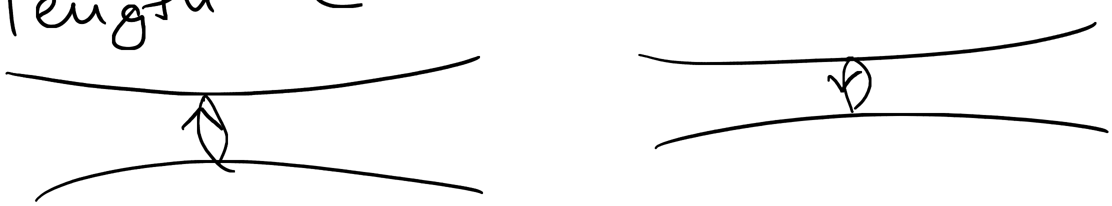
In this lecture we show that $Z_M(s)$ extends to an <u>entire</u> function of $s \in \mathbb{C}$.

We will do this using <u>transfer operators</u> and <u>Fredholm determinants</u>.

<u>Example</u>: hyperbolic cylinder

$\mathbb{R}_r \times (\mathbb{R}/\ell\mathbb{Z})_\theta$, $g = dr^2 + \cosh^2 r \, d\theta^2$

2 primitive closed geodesics of length ℓ because direction matters:



$$Z_M(s) = \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell}\right)^2$$

From here we see that $Z_M(s)$ is <u>entire</u> and its zeros are when $e^{-(s+k)\ell} = 1$, i.e.

$s = \frac{2\pi i}{\ell} p - k$, $p \in \mathbb{Z}$, $k \in \mathbb{N}_0$.

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Relation of closed geodesics to words in Γ:

Assume that $(x, v) \in SM$ gives a closed geodesic of period $T>0$, i.e. $\boxed{\varphi^T(x,v) = (x,v).}$

Let us lift it: if $\tilde{\pi} : S\mathbb{H}^2 \to SM$ is the projection map then choose $(y, w) \in S\mathbb{H}^2$ such that

$$\boxed{\tilde{\pi}(y,w) = (x,v)}$$

Denoting by φ^t the geodesic flows on SM and on $S\mathbb{H}^2$, we then have

$$\tilde{\pi}(\varphi^T(y,w)) = \varphi^T(x,v) = (x,v) = \tilde{\pi}(y,w).$$

Thus $\exists\, \gamma \in \Gamma \setminus \{I\}$ such that (with the standard action of γ on $S(\mathbb{H}^2)$)

$$\boxed{\varphi^T(y,w) = \gamma.(y,w).}$$

<u>Example:</u> hyperbolic cylinder

Say for simplicity that Γ is generated by $\gamma_1 = \begin{pmatrix} e^{\ell/2} & 0 \\ 0 & e^{-\ell/2} \end{pmatrix}$, $\gamma_1 . z = e^{\ell} . z$

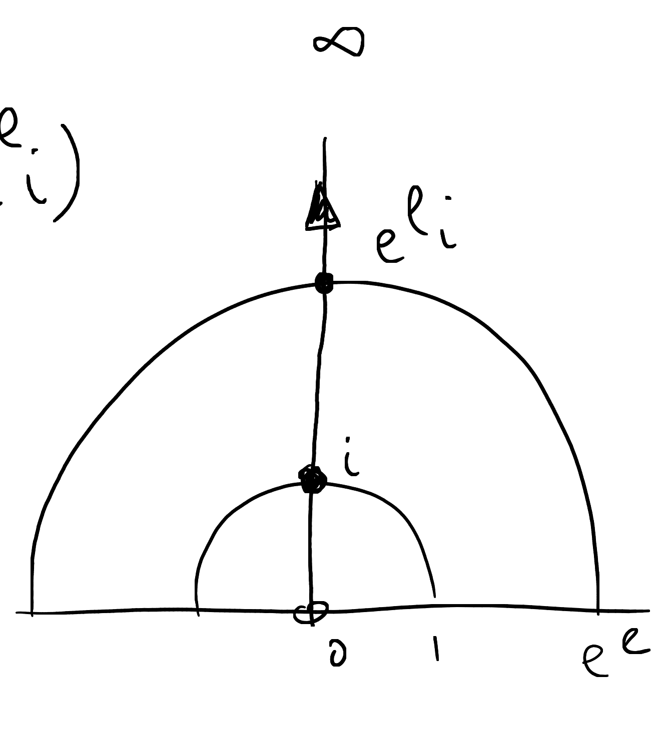
$\gamma_1.(y,w) = (e^\ell y, e^\ell w)$.

Then we take $T = \ell$, $\gamma = \gamma_1$,

$(y,w) = (i, i)$:

$\varphi^T(y,w) = (e^\ell i, e^\ell i)$

$= \gamma_1.(y,w)$.



Coming back to the general case:

$(y,w) \in SH^2$, $\gamma \in \Gamma \setminus \{I\}$, $T > 0$,

$$\boxed{\varphi^T(y,w) = \gamma.(y,w)}$$

This describes all closed geodesics on M up to the following transformations which would give the same geodesic:

- $(y,w) \mapsto \varphi^s(y,w)$, γ same
  (then $\tilde{\pi}(\varphi^s(y,w)) = \varphi^s(\tilde{\pi}(y,w))$ is a different point on the same closed geodesic)
- $(y,w) \mapsto \tilde{\gamma}.(y,w)$ for some $\tilde{\gamma} \in \Gamma$
  (taking a different preimage in SH^2 of the same point $(x,v) \in SM$)

and $\gamma \mapsto \tilde{\gamma} \cdot \gamma \cdot \tilde{\gamma}^{-1}$

Now, since $\gamma \in \Gamma \setminus \{I\}$

we can write

$\gamma = \gamma_{\vec{a}}$ for some $\vec{a} = a_1 \ldots a_n \in W^n$.

Recall that γ here is defined up to conjugation by an element of Γ.

Then we can choose $\gamma_{\vec{a}}$

so that $\boxed{a_1 \neq \bar{a}_n.}$

Indeed, if $a_1 = \bar{a}_n$, then

$$\gamma = \gamma_{a_1} \gamma_{a_2 \ldots a_{n-1}} \gamma_{a_n} = \gamma_{a_1} \gamma_{a_2 \ldots a_{n-1}} \gamma_{a_1}^{-1}$$

is conjugate to $\gamma_{a_2 \ldots a_{n-1}}$

and we can repeat the process...

The element $\gamma_{\vec{a}}$ is <u>hyperbolic</u>:

$$\gamma_{\vec{a}}(D_{a_1}) \subset D_{a_1}, \quad \gamma_{\vec{a}}^{-1}(D_{\bar{a}_n}) \subset D_{\bar{a}_n},$$

so $\gamma_{\vec{a}}$ has an attractive / repulsive fixed pt $\gamma_{+,\vec{a}} \in D_{a_1}$, $\gamma_{-,\vec{a}} \in D_{a_n}$.

In fact, $\forall j \geq 0$

$$\vartheta_{+,\vec{a}} \in \gamma_{\vec{a}}^{j}(D_{a_1})$$

$$= D_{\underbrace{\vec{a}\,\vec{a}\ldots\vec{a}}_{\text{concatenated } j \text{ times}}\, a_1},$$

so $\vartheta_{+,\vec{a}} \in \Lambda_\Gamma$.

Similarly $\vartheta_{-,\vec{a}} \in \Lambda_\Gamma$.

And $\varphi^T(y,w) = \gamma_{\vec{a}} \cdot (y,w)$

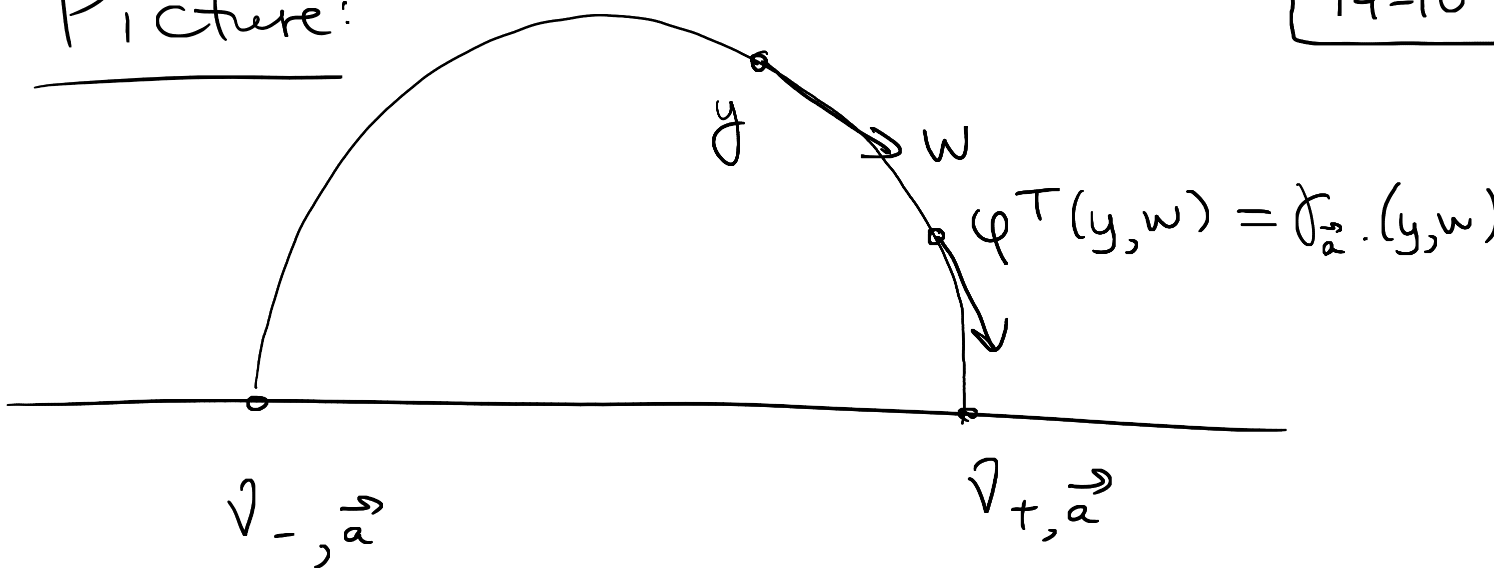
$$\Updownarrow$$

(y,w) lies on the geodesic from $\vartheta_{-,\vec{a}}$ to $\vartheta_{+,\vec{a}}$

and $\boxed{e^T = \gamma_{\vec{a}}'(\vartheta_{-,\vec{a}})}$

Indeed, we can conjugate $\gamma_{\vec{a}}$ to $\begin{pmatrix} e^{\ell/2} & 0 \\ 0 & e^{-\ell/2} \end{pmatrix}$ in which case this is straightforward to check, with the repulsive fixed point $\vartheta_{-,\vec{a}} \sim 0$ and $\partial_z(e^\ell z)\big|_{z=0} = e^\ell$.

Picture:



Thus we see that

closed geodesics on $M = \Gamma \backslash \mathbb{H}^2$

are bijective to

closed words

$\vec{a} = a_1 \ldots a_n \in \mathcal{W}^n, \quad a_n \neq \overline{a}_1$

modulo conjugation of the corresponding $\gamma_{\vec{a}}$ (in Γ)

We can check that

2 closed words give conjugate $\gamma_{\vec{a}}$

iff they are cyclic permutations of each other

e.g. $\gamma_{a_1 \ldots a_n}$ conjugate to $\gamma_{a_2 \ldots a_n a_1}$.

Finally, a closed geodesic is primitive iff the corresponding word is primitive i.e. it's not a power of another word.

More precisely / to recap:

<u>Defn.</u> • A closed word is $a_1 \dots a_n$ s.t. $a_1, \dots, a_n \in \mathcal{A}$ and $\bar{a}_1 \neq a_2, \bar{a}_2 \neq a_3, \dots, \bar{a}_{n-1} \neq a_n, \bar{a}_n \neq a_1$.

• A closed word $\vec{a} = a_1 \dots a_n$ is <u>primitive</u>, if there exists no $l < n$ s.t. $a_{j+l \,(\mathrm{mod}\, n)} = a_j \;\forall j$
(i.e. not a power of a shorter word)
(e.g. 121 primitive, 1212 not primitive)

• Two closed words $a_1 \dots a_n$, $b_1 \dots b_n$ are equivalent if $\exists l \;\forall j$
$a_j = b_{(j+l) \,\mathrm{mod}\, n}$ (e.g. $123 \sim 312 \sim 231$ but $123 \not\sim 132$)

Then we have

Thm The set of primitive (oriented) closed geodesics on $M = \Gamma \backslash \mathbb{H}^2$ is in one-to-one correspondence with the set of equivalence classes of primitive closed words $\gamma_{\vec{a}}$ in the group Γ.

Here the length ℓ of a geodesic is related to $\vec{a}$ by the formula

$$e^{\ell} = \gamma_{\vec{a}}'(\eta_{-,\vec{a}})$$

where $\eta_{-,\vec{a}}$ is the repulsive fixed point of $\gamma_{\vec{a}}$.

$\downarrow$

Example 1: hyperbolic cylinder

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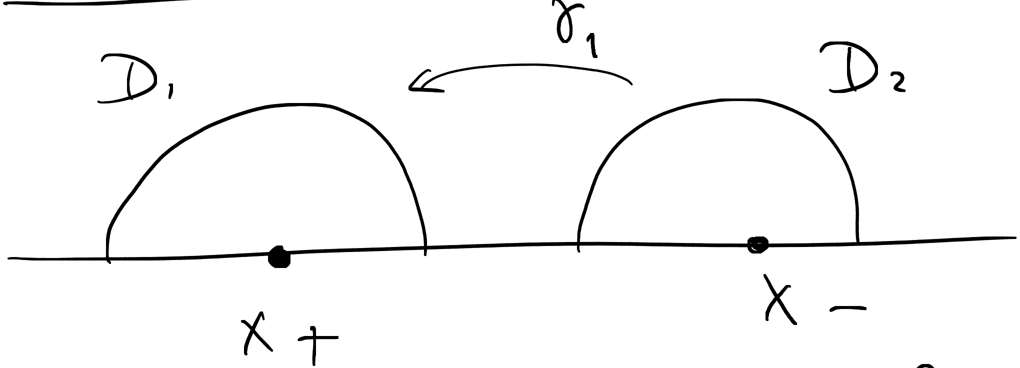
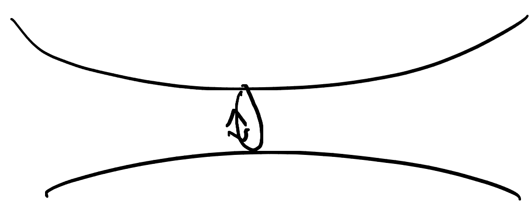
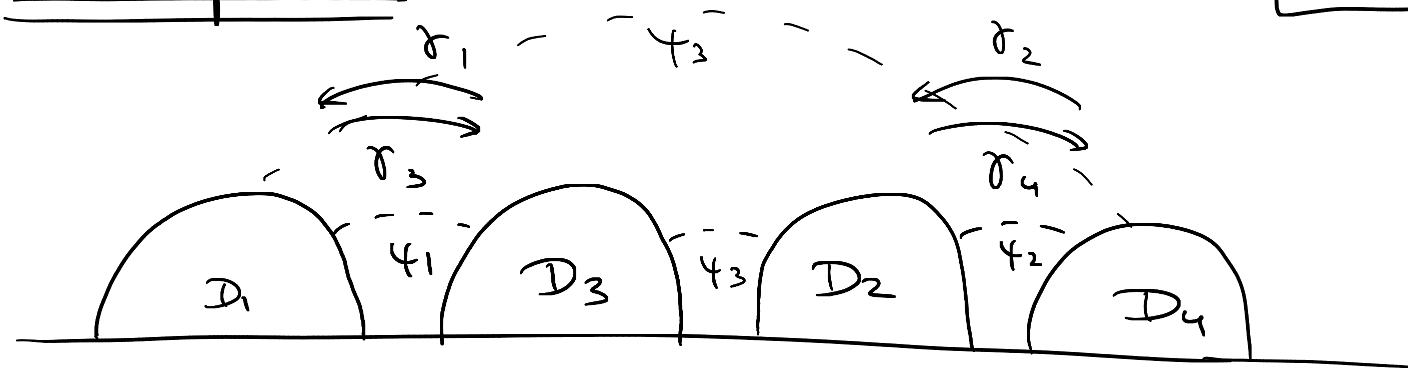


$x_\pm$ fixed points of γ_1

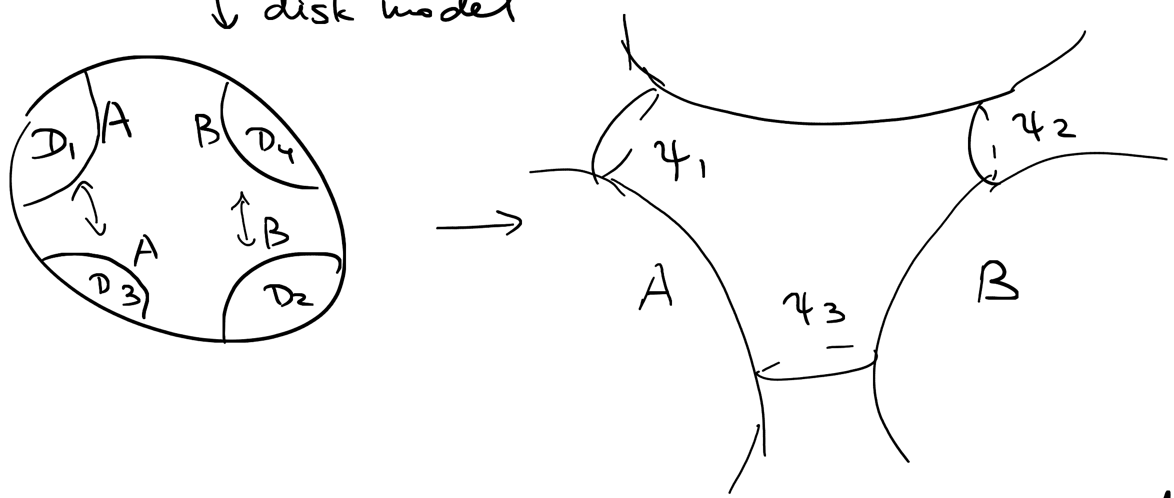
There are 2 primitive closed words: 1 and 2. These correspond to the 2 primitive closed geodesics on M:



Example 2: 3-funnel surface



↓ disk model



One can check that the closed geodesics ψ_1, ψ_2, ψ_3 correspond to the words (for a specific choice of orientation on ψ_j)

$$\psi_1 \sim 1$$
$$\psi_2 \sim 2$$
$$\psi_3 \sim 12$$

<u>Rmk</u>. This all agrees with geometrical considerations: Γ = fundamental group and curvature <0 $\Rightarrow$ exactly 1 closed geodesic in each $\underbrace{\text{free}}_{\text{nontrivial}}$ <u>free homotopy class</u> $= \Gamma/\text{conjugation}$

§14.2. Determinants of transfer operators

Recall the transfer operator

$\mathcal{L}_s : C^0(\Lambda_\Gamma) \circlearrowleft$,

$$\mathcal{L}_s f(x) = \sum_{\substack{a \in \mathcal{A} \\ a \neq \bar{b}}} f(\gamma_a(x))\, \gamma_a'(x)^s, \quad x \in \Lambda_\Gamma \cap D_b,\ b \in \mathcal{A}.$$

We now define it on a <u>different space</u>:

Denote $\mathbb{D} = \bigsqcup_{a \in \mathcal{A}} D_a$ (union of the original Schottky disks)

here $\mathbb{D} \subset \mathbb{C}$.

Define $\mathcal{H}(\mathbb{D}) = \{ f \in L^2(\mathbb{D})$:

wrt Lebesgue measure on $\mathbb{C}$

f is <u>holomorphic</u> on $\mathbb{D}^\circ \}$

One can show that $\mathcal{H}(\mathbb{D}) \subset L^2(\mathbb{D})$ is a closed subspace w.r.t. L^2 norm and thus $\mathcal{H}(\mathbb{D})$ is a Hilbert space.

Trace class operators (a very brief intro)

If $\mathcal{H}$ is a Hilbert Space, we say that a compact operator $A : \mathcal{H} \to \mathcal{H}$ is **trace class**, if

$\sum_j \sigma_j(A) < \infty$ where $\sigma_1(A), \sigma_2(A) \ldots$ are the singular values of A:

$\text{Spectrum}(A^* A) = \{\sigma_j(A)^2\}$

$\mathcal{L}_1(\mathcal{H}) :=$ space of all trace class operators

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The expression

$$\|A\|_{Tr} = \sum_j \sigma_j(A)$$

defines a norm, called the trace norm.

Properties:

- $(\mathcal{L}_1(\mathcal{H}), \|\cdot\|_{Tr})$ is a Banach Space and finite rank operators are dense in it
- Ideal property: if $A \in \mathcal{L}_1(\mathcal{H})$ and $B: \mathcal{H}\circlearrowleft$ is bounded then $AB, BA \in \mathcal{L}_1(\mathcal{H})$ and $\|AB\|_{Tr}, \|BA\|_{Tr} \leq \|A\|_{Tr} \cdot \|B\|_{\mathcal{H}\circlearrowleft}$

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- If $A \in \mathcal{L}_1(H)$, we can define its <u>trace</u>

$\operatorname{tr} A \in \mathbb{C}$:

$\operatorname{tr} A$ defined the usual way on finite rank A & continuous w.r.t. $\|\cdot\|_{Tr}$

- If $A \in \mathcal{L}_1(\mathcal{H})$, we can define the <u>determinant</u>

$\det(I+A) \in \mathbb{C}$

which is the usual det when A has finite rank & continuous w.r.t. $\|\cdot\|_{TR}$

- $\det(I+A) = 0 \iff$

$\iff I+A$ is <u>not</u> invertible $\mathcal{H} \to \mathcal{H}$

Coming back to $\mathcal{L}_s$,

for $s \in \mathbb{C}$ define

$\mathcal{L}_s : \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D})$ by

$$\mathcal{L}_s f(z) = \sum_{\substack{a \in \mathcal{A} \\ a \neq \bar{b}}} f(\gamma_a(z))[\gamma_a'(z)]^s, \quad z \in D_b, \; b \in \mathcal{A}.$$

Here $[\gamma_a'(z)]^s = \exp(s \log \gamma_a'(z))$

and $\log \gamma_a'(z)$ is well-defined

by requiring $\log \gamma_a'(x) \in \mathbb{R}$

when $x \in I_b = D_b \cap \mathbb{R}$.

This works since for $a \neq \bar{b}$ we

have $\gamma_a(D_b) = D_{ab} \subset D_a$.

Now we can present

Thm The operator $\mathcal{L}_s$ is trace class on $\mathcal{H}(\mathbb{D})$ for all $s \in \mathbb{C}$ and for $\operatorname{Re} s \gg 1$ we have

$$\boxed{Z_M(s) = \det(I - \mathcal{L}_s)}$$

Selberg zeta fn

This gives the holomorphic extension of Z_M to $\mathbb{C}$, since $\det(I - \mathcal{L}_s)$ is holomorphic in $s \in \mathbb{C}$.

We will not give the proof, see Borthwick, Thm 15.10.

But here are a few ingredients:

① If $\Omega \subset \mathbb{C}$ is open and $\gamma : \Omega \to \Omega$ is a holomorphic map such that $\boxed{\gamma(\Omega) \subset K}$ for some compact $K \subset \Omega$, and we define the operator $\gamma^* : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$,

$$\gamma^* f(z) = f(\gamma(z)),$$

$$\mathcal{H}(\Omega) = \{ f \in L^2(\Omega; \text{Lebesgue}) : f \text{ is holomorphic on } \Omega \}$$

Then $\gamma^* : \mathcal{H}(\Omega) \circlearrowleft$ is trace class.

Moreover, if γ has a unique fixed point $z_0 \in \Omega$, with $\gamma'(z_0) \neq 1$, then

$$\boxed{\operatorname{tr} \gamma^* = \frac{1}{1 - \gamma'(z_0)}}$$

(Lefschetz fixed point formula)

Proof (sketch)

Take a contour $\mathcal{C} \subset \Omega$ enclosing K:

Then $\forall z \in \Omega$,

$\gamma(z) \in K$, so by the Cauchy Integral Formula we have

$$\gamma^* f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{w - \gamma(z)}\, dw.$$

That is, $\gamma^* = \frac{1}{2\pi i} \oint_{\mathcal{C}} A_w \, dw$

as operators on $\mathcal{H}(\Omega)$, where

$A_w : \mathcal{H}(\Omega) \circlearrowleft$,

$$A_w f(z) = \frac{f(w)}{w - \gamma(z)}, \quad z \in \Omega, \ w \in \mathcal{C}.$$

A_w is <u>rank 1</u> and thus trace class and the $\oint_{\mathcal{C}}$ converges in trace class norm. So γ^* is trace class.

To compute the trace, note that

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$\forall w \in \mathcal{C}, \quad A_w = u_w \otimes v_w$

in the sense that $A_w f = u_w(f) \cdot v_w \quad \forall f \in \mathcal{H}(\Omega)$

where $u_w : \mathcal{H}(\Omega) \to \mathbb{C}, \quad v_w \in \mathcal{H}(\Omega)$

are given by

$$u_w(f) = f(w), \quad v_w(z) = \frac{1}{w - \gamma(z)}.$$

Then $\operatorname{tr} A_w = u_w(v_w) = \dfrac{1}{w - \gamma(w)}$.

Integrating over $\mathcal{C}$, we get

$$\operatorname{tr} \gamma^* = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{dw}{w - \gamma(w)}$$

and it remains to use the Residue Thm

to see that $\operatorname{tr} \gamma^* = \dfrac{1}{1 - \gamma'(z_0)}$. $\square$

(2) Using (1), we can check that $\mathcal{L}_s : \mathcal{H}(D) \circlearrowleft$ is trace class (as $\forall a \neq \bar{b}$, $\gamma_a(D_b) \Subset D_a$) and compute the <u>trace</u> of $\mathcal{L}_s^k$ for any $k \geq 1$ using the Lefschetz fixed point formula. The result features the fixed points of $\gamma_{\vec{a}}$ for closed words $\vec{a} = a_1 \ldots a_k$, which are related to closed geodesics on M by the discussion in §14.1.

To deal with the <u>determinant</u>, we write $\forall z \in \mathbb{C}$ small enough

$$\log \det (I - z\mathcal{L}_s) = \operatorname{tr} \log (I - z\mathcal{L}_s)$$

(z small)

$$= -\operatorname{tr} \sum_{k \geq 1} \frac{(z\mathcal{L}_s)^k}{k} = -\sum_{k \geq 1} \frac{z^k}{k} \operatorname{tr} \mathcal{L}_s^k$$

already computed

And for $\operatorname{Re} s \gg 1$, can take $z = 1$. $\square$

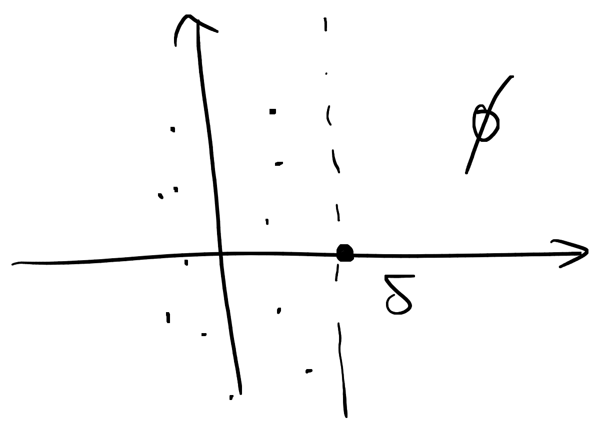
§14.3. More on zeta functions

(NO PROOFS AT ALL HERE)

Let (M,g) be a convex co-compact hyperbolic surface which is <u>not</u> a hyperbolic cylinder.

Here are some <u>basic facts</u> about the Selberg zeta function $Z_M(s)$ (or rather, its extension to $s \in \mathbb{C}$):

- Z_M has a simple zero at $\boxed{s=\delta}$ (note: $Z_M(s) = \det(I - \mathcal{L}_s)$, $(I-\mathcal{L}_\delta)^* \mu = \mu$ where μ is the Patterson–Sullivan measure)
- Z_M has no other zeros in $\mathrm{Re}\, s \geq \delta$:



- We call zeros of $Z_M(s)$ the <u>resonances</u> of M

Using these facts, one can show | 18.118 14-26

Prime Geodesic Thm

Let $\mathcal{N}_M(R) = \{$number of primitive oriented closed geodesics on M of period $\le R\}$.

Then as $R \to \infty$,

$$\mathcal{N}_M(R) = \frac{e^{\delta R}}{\delta R}\left(1 + o(1)\right).$$

For the proof, see e.g. Borthwick, Thm. 14.20.

(Note: still works if M is compact in which case $\delta = 1$)

Naud 2005: $\exists\ \varepsilon = \varepsilon(M) > 0$ s.t. $\mathcal{N}_M(R) = li(e^{\delta R}) + O(e^{(\delta - \varepsilon)R})$

where $li(x) = \int_2^x \frac{dt}{\log t}$.

Bourgain-D '17: can take $\varepsilon = \varepsilon(\delta)$ only (uses additive combinatorics)

What about general geodesic flows on negatively curved manifolds?

<u>Thm</u> [Giulietti - Liverani - Pollicott 2012]

Let $\varphi^t : X \circlearrowleft$ be an <u>Anosov flow</u>.

Then the <u>Ruelle zeta function</u>

$$\zeta_R(s) = \prod_{\ell \in \mathcal{L}_{\varphi^t}} (1 - e^{-s\ell}), \quad \operatorname{Re} s \gg 1,$$

$\underbrace{\mathcal{L}_{\varphi^t}}$ lengths of primitive closed orbits of φ^t

admits a <u>meromorphic continuation</u> to $s \in \mathbb{C}$.

<u>First pole</u>: $\delta = h_{top}(\varphi^t)$
topological entropy

and a version of Prime Orbit Thm holds

2013
D-Zworski: another proof, writing $Z(s) =$ "det $(P - s)$" where P "=" $\mathcal{L}_V$, Lie derivative w.r.t. V, $\varphi^t = e^{tV}$, on differential forms.

This is not trace class but one can make sense of det using the Atiyah-Bott-Guillemin trace formula, Hörmander's propagation of singularities, and Melrose's radial estimates

D-Guillarmou 2014:
noncompact case, e.g. "convex co-compact" variable negative curvature manifolds

There are also relations to topology: multiplicity of $s=0$ as a singularity of $Z(s)$ is related to Betti numbers...