# 18.118 "Introduction to Chaotic Dynamics"

This is a course in (a small subset of) **Dynamical Systems**

What do we study in Dynamical Systems?

Let's say $X$ is a set

(will typically be a metric space, often will be a manifold → review those!)

and $T: X \to X$ is a map

(will typically be continuous / smooth often but not always will be invertible)

Define the n-th iterate

$$T^n = T \circ \cdots \circ T \quad n \text{ times}, \quad n \in \mathbb{N}_0$$

i.e. $T^0 = Id$, $T^{n+1} = T \circ T^n$

A basic goal is:

For $x \in X$, study the trajectory

$T^n(x) \in X$ as $n \to \infty$.

# §1.1. An example: irrational shift on the circle

Let us take

$$X = \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$$

and the map $T : X \to X$ given by

$$T([x]) = (x + r) \bmod \mathbb{Z}, \quad [x] \in \mathbb{R}/\mathbb{Z}$$

where $r \in \mathbb{R} \setminus \mathbb{Q}$ is a fixed <u>irrational</u> number.

We want to study the statistics of the trajectory $T^n(x)$ as $n \to \infty$.

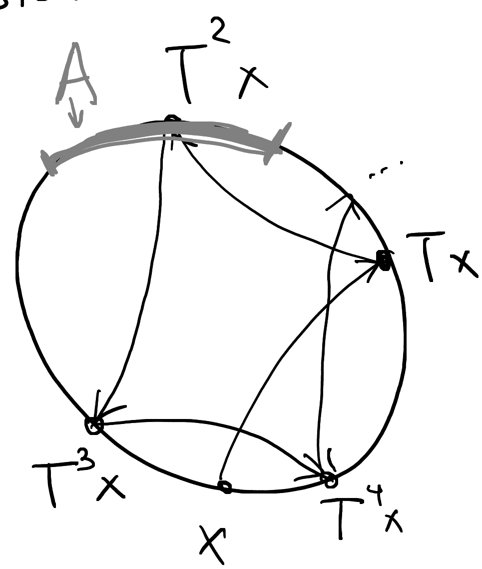
Let us fix some set $A \subset \mathbb{S}^1$.

We can look at the expression

$$\frac{1}{n} \# \{ j = 0, \dots, n-1 : T^j(x) \in A \}$$

and take the limit as $n \to \infty$.

(What <u>proportion of time</u> does the trajectory spend in $A$?)



The set $A$ could be quite bad, so actually we replace this with a different question.

Let us rewrite

$$\frac{1}{n}\#\{j=0,\dots,n-1 : T^j(x)\in A\} =$$

$$= \frac{1}{n}\sum_{j=0}^{n-1} 1_A(T^j(x))$$

where $1_A(x) = \begin{cases} 1, & x\in A \\ 0, & x\notin A \end{cases}$

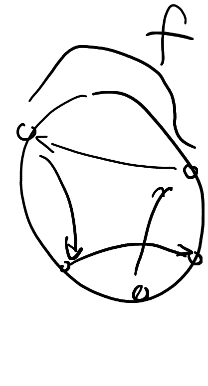
is the <u>indicator function</u> of $A$.

This prompts

<u>Defn</u> Let $f : X \to \mathbb{R}$ be a function.

The <u>ergodic average</u> at time $n\in\mathbb{N}$ of $f$ w.r.t. $T: X\to X$ is the function $\langle f\rangle_n : X\to\mathbb{R}$ defined by

$$\langle f\rangle_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} f(T^j(x))$$




(averaging $f$ over the trajectory)

We now prove

**Thm.** (Unique ergodicity of irrational shift)

Assume that $f \in C^0(S^1)$. Then
continuous functions

$$\langle f \rangle_n(x) \xrightarrow{n\to\infty} \int_0^1 f(x)\,dx \quad \text{uniformly in } x \in [0,1]$$

(equidistribution of the trajectories of $T$ w.r.t. Lebesgue measure)

**Rmk** This implies in particular that $\forall x \in S^1$, the trajectory $\{T^n(x) \mid n \geq 0\}$ is dense in $S^1$.

Indeed, let $A \subset S^1$ be open nonempty.

Take $f \in C^0(S^1)$: $\operatorname{supp} f \subset A$ and $c = \int_0^1 f(x)\,dx > 0$. ($\operatorname{supp} f \overset{\text{def}}{=}$ the closure of $\{x : f(x) \neq 0\}$)

Then $\langle f \rangle_n(x) \xrightarrow{n\to\infty} c > 0$, so $\exists n : \langle f \rangle_n(x) \neq 0$. Since $\langle f \rangle_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} f(T^j(x))$, we see that $\exists j : f(T^j(x)) \neq 0$. Then $T^j(x) \in A$, so the trajectory of $x$ intersects $A$.

To prove the Thm, we use the following general fact from functional analysis:

<u>Lemma</u> Assume $\mathcal{X}, \mathcal{Y}$ are normed vector spaces and $B_n : \mathcal{X} \to \mathcal{Y}$ is a sequence of <u>bounded</u> linear operators such that

(1) There exists a dense set $S \subset \mathcal{X}$ such that $B_n f \to Bf$ in $\mathcal{Y}$ for all <u>$f \in S$</u>, where $B : \mathcal{X} \to \mathcal{Y}$ is some bounded linear operator, and

(2) $\exists\, C : \forall n, \|B_n\|_{\mathcal{X} \to \mathcal{Y}} \leq C$

Then $B_n f \to Bf$ in $\mathcal{Y}$ for all <u>$f \in \mathcal{X}$</u>.

## Proof of Lemma



Let $f \in \mathcal{X}$. Since $S$ is dense in $\mathcal{X}$, we can find a sequence

$$f_k \in S, \quad f_k \to f \text{ in } \mathcal{X}.$$

For each $k, n$ we can estimate

$$\|B_n f - Bf\|_Y \leq \|B_n f_k - Bf_k\|_Y + \|B_n f - B_n f_k\|_Y + \|Bf - Bf_k\|_Y$$

$$\leq \|B_n f_k - Bf_k\|_Y + 2C\|f - f_k\|_{\mathcal{X}}.$$

Fixing $k$ and taking $\limsup_{n\to\infty}$, get

$$\limsup_{n\to\infty} \|B_n f - Bf\|_Y \leq 2C\|f - f_k\|_{\mathcal{X}} \qquad (*)$$

since $f_k \in S$ and thus $B_n f_k \to Bf_k$ in $Y$.

Since $(*)$ is true for all $k$, take the limit in $k$ to get

$$\limsup_{n\to\infty} \|B_n f - Bf\|_Y \leq 0 \Rightarrow B_n f \to Bf \text{ in } Y.$$

$\square$

We can now give

<u>Proof of Thm</u> We use the normed vector space $C^0(S^1)$ with the sup-norm.

Consider the operator $B_n : C^0(S^1) \circlearrowleft$ given by $B_n f(x) = \langle f \rangle_n (x) = \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(x))$.

Note that $\|B_n\|_{C^0 \circlearrowleft} \leq 1$ since

$$\sup_x |\langle f \rangle_n (x)| \leq \sup_x |f(x)|.$$

So by the Lemma it suffices to show that $\langle f \rangle_n (x) \xrightarrow[n\to\infty]{} \int_0^1 f(x)\,dx$ in $C^0$ for $f$ in some dense subset $S \subset C^0$.

We choose $S = \{$trigonometric polynomials$\}$
$= \{$finite linear combinations of $e^{2\pi i \ell x}$, $\ell \in \mathbb{Z}\}$

($S$ is dense in $C^0$ by the theory of Fourier series / Stone-Weierstraß Thm)

Since $f \mapsto \langle f \rangle_n$ is linear, it is then enough to show that $\forall \ell \in \mathbb{Z}$ and $e_\ell(x) = e^{2\pi i \ell x}$, we have

$$\langle e_\ell \rangle_n (x) \longrightarrow \int_0^1 e_\ell(x)\,dx = \begin{cases} 1, & \text{if } \ell = 0 \\ 0, & \text{if } \ell \neq 0. \end{cases}$$

$\underline{\underline{\ell = 0}}$: immediate since $e_\ell \equiv 1$, so $\langle e_\ell \rangle_n \equiv 1$ as well

$\underline{\underline{\ell \neq 0}}$: we compute (using that $T(x) = x + r \mod \mathbb{Z}$)

$$\langle e_\ell \rangle_n (x) = \frac{1}{n} \sum_{j=0}^{n-1} e_\ell(T^j(x))$$

$$= \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i \ell (x + jr)}$$

$$= \frac{e^{2\pi i \ell x}}{n} \sum_{j=0}^{n-1} \left(e^{2\pi i \ell r}\right)^j = \frac{e^{2\pi i \ell x}}{n} \cdot \frac{e^{2\pi i \ell r n} - 1}{e^{2\pi i \ell r} - 1}$$

$$\downarrow_{n \to \infty}$$

$$0$$

since $e^{2\pi i \ell r} \neq 1$ as $r$ is irrational. $\square$

# §1.2. A bit about flows

In addition to maps, we will study flows.

Assume $X$ is a manifold (for now, compact)

and $V \in C^\infty(X; TX)$ is a vector field on $X$.

We can define the flow of $V$,

$$e^{tV} : X \to X, \quad t \in \mathbb{R},$$

as the solution of an ODE:

if $x_0 \in X$ then $\gamma(t) = e^{tV}(x)$

satisfies the initial value problem

$$\begin{cases} \dot{\gamma}(t) = V(\gamma(t)) \leftarrow \text{a tangent vector to } X \text{ at the point } \gamma(t) \\ \gamma(0) = x_0 \end{cases}$$

We have $e^{(t+s)V} = e^{tV} e^{sV}$ (one-parameter group)

and $\forall t \in \mathbb{R}$, $e^{tV} : X \circlearrowleft$ is a $C^\infty$ diffeomorphism

An important family of examples is given by geodesic flows:

- $(M,g)$ compact Riemannian manifold
- $X = SM = \{(x,v) \in TM : |v|_g = 1\}$ is the sphere bundle of $M$
- $e^{tV} : X \to X$ is the geodesic flow on $M$: if $(x_0, v_0) \in TM$ then $e^{tV}(x_0, v_0) = (\gamma(t), \dot\gamma(t))$ where $\gamma : \mathbb{R} \to M$ is the geodesic such that $\gamma(0) = x_0$, $\dot\gamma(0) = v_0$

One goal of the first part of this course is:

**Thm** If $(M,g)$ has negative sectional curvature then $e^{tV}$ is ergodic with respect to the Liouville measure $\mu$: for all $f \in L^1(SM)$ and for $\mu$-almost every $y \in SM$ we have

$$\frac{1}{t} \int_0^t f(e^{tV}(y))\,dt \xrightarrow[t \to \infty]{} \int_{SM} f\,d\mu.$$

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# §1.3. Review of measure theory

Assume that $X$ is a complete metric space.

We use <u>measure theory</u>: (18.125) please review the basics!

- Borel $\sigma$-algebra $\mathcal{B}(X) \subset$ the set of all subsets of $X$ ("measurable" sets; the smallest $\sigma$-algebra containing all open sets)
- Probability measure: a map
  $$\mu : A \in \mathcal{B}(X) \mapsto \mu(A) \geq 0$$
  which is <u>countably additive</u> and satisfies <u>$\mu(X) = 1$</u>
- Lebesgue Integral: if $f : X \to [0, \infty]$ is <u>Borel measurable</u> (i.e. $\forall A \subset [0,\infty]$ Borel, $f^{-1}(A)$ is Borel; $f$ continuous $\Rightarrow$ $f$ measurable)

then can define
$$\int_X f \, d\mu \in [0, \infty]$$

If $f : X \to \overline{\mathbb{R}} = [-\infty, \infty]$ is $\mu$-measurable

and $\int_X |f| \, d\mu < \infty$

then can define $\int_X f \, d\mu \in \mathbb{R}$

- $\int_X |f| = 0 \iff f = 0$ $\mu$-almost everywhere,

  i.e. $\mu(\{x \in X \mid f(x) \neq 0\}) = 0$.

- The spaces $L^p$: (for $1 \leq p < \infty$)

$$L^p(X, \mu) = \frac{\{f : X \to \mathbb{R} \ \mu\text{-measurable and } \int_X |f|^p \, d\mu < \infty\}}{\{f : X \to \mathbb{R} \ \mu\text{-measurable and } f = 0 \ \mu\text{-almost everywhere}\}}$$

$L^p(X, \mu)$ is a Banach space (complete normed vector space)

with the norm $\|f\|_{L^p} = \left( \int_X |f|^p \, d\mu \right)^{1/p}$

$L^2(X, \mu)$ is a Hilbert space with the inner product $\langle f, g \rangle_{L^2} := \int_X f \cdot g \, d\mu$

Sometimes we will use complex valued functions $f: X \to \mathbb{C}$, then

$$\langle f, g\rangle_{L^2(X,\mu)} = \int_X f \cdot \bar{g} \, d\mu .$$

**Note:** $L^2(X,\mu)$ is <u>separable</u>, i.e. it has a <u>Hilbert basis</u> (a $\leq$ countable orthonormal system $(e_j)_{j\in\mathbb{N}}$ s.t. $\mathrm{Span}(e_j)$ is dense), if $X$ compact (or 2nd countable ...)

We now discuss <u>weak convergence</u>:

**Defn** Let $\mu_n$ be a sequence of probability measures on $X$. We say that $\mu_n$ <u>converges weakly</u> to some probability measure $\mu$, if

$$\int_X f \, d\mu_n \xrightarrow[n\to\infty]{} \int_X f \, d\mu$$

for every $f \in C^0(X)$ bounded.

We have the following

**Compactness Thm**

Assume $X$ is a compact metric space and $\mu_n$ is any sequence of probability measures. Then $\exists$ a subsequence $\mu_{n_\ell}$ which converges to some $\mu$ weakly.

For the proof, we need 2 facts:

**Thm [Continuous Linear extension]**

If $\mathcal{X}$ is a normed vector space,
$S \subset \mathcal{X}$ is a dense subspace,
$\mathcal{Y}$ is a Banach space,
and $B : S \to \mathcal{Y}$ is bounded
(w.r.t. the norms $\|\cdot\|_{\mathcal{X}}, \|\cdot\|_{\mathcal{Y}}$)

Then there exists unique $\tilde{B} : \mathcal{X} \to \mathcal{Y}$ which is bounded (in fact, $\|\tilde{B}\|_{\mathcal{X}\to\mathcal{Y}} = \|B\|_{S\to\mathcal{Y}}$) and satisfies $\tilde{B}|_S = B$

Sketch of proof: Take $f \in \mathcal{X}$ and approximate it: $f_n \in S$, $f_n \to f$ in $\mathcal{X}$. Then $Bf_n$ is a Cauchy sequence in $\mathcal{Y}$. Define $Bf := \lim_{n\to\infty} Bf_n$ .... $\square$

**Thm** [Riesz Representation for $C^0$]

Let $X$ be a compact metric space and $F : C^0(X) \to \mathbb{R}$ is a map such that:

($C^0$: continuous functions)

1. $F$ is linear
2. $f \geq 0 \Rightarrow F(f) \geq 0$
3. $F(\mathbb{1}) = 1$ where $\mathbb{1} \in C^0(X)$ is a constant function

Then there exists unique probability measure $\mu$ on $X$ such that

$$F(f) = \int_X f \, d\mu \quad \text{for all } f \in C^0(X).$$

**Proof**: see for instance

Stroock, Essentials of Integration Theory for Analysis, Thm. 8.2.16

or

Rudin, Real and Complex Analysis, Thm. 2.14

Either of these texts can also be used as an introduction to measure theory & Lebesgue integral

We can now give

**Proof of Compactness Thm (sketch)**

Since $X$ is compact, the space $C^0(X)$ is separable (with sup-norm as usual).

Thus there exists a sequence

$f_k \in C^0(X)$, $k \in \mathbb{N}$, such that

$S := \operatorname{Span}(f_k)$ is dense in $C^0(X)$.

For each $k$, the sequence $n \mapsto \int_X f_k \, d\mu_n$ is bounded (by $\sup |f_k|$).

Using a diagonal argument (similarly to the proof of Arzelà-Ascoli Thm) we construct a subsequence $\mu_{n_\ell}$ such that

$\lim_{\ell \to \infty} \int_X f_k \, d\mu_{n_\ell}$ exists for each $k$.

Then $\lim_{\ell \to \infty} \int_X f \, d\mu_{n_\ell}$ exists for all $f \in S$.

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Define $B : \underset{C^\circ(X)}{\underset{\cap}{S}} \to \mathbb{R}$ by

$$Bf = \lim_{\ell\to\infty} B_\ell f, \quad f \in S \quad \text{where}$$

$$B_\ell f = \int_X f \, d\mu_{n_\ell}.$$

Since each $B_\ell$ is linear and bounded uniformly in $\ell$ ($|B_\ell f| \leq \sup|f| = \|f\|_{C^\circ}$), we see that $Bf : S \to \mathbb{R}$ is also linear and bounded ($|Bf| \leq \|f\|_{C^\circ}$).

By Continuous Linear Extension, we can extend $B$ to $\tilde{B} : C^\circ(X) \to \mathbb{R}$ linear, bounded. And by Lemma from §1.1, we have $B_\ell f \xrightarrow[\ell\to\infty]{} \tilde{B} f$ for <u>all</u> $f \in C^\circ(X)$

Now, passing to the limit in $\ell$, we see:

1. $f \geq 0 \Rightarrow \tilde{B} f \geq 0$
2. $\tilde{B}(\mathbb{1}) = 1$.

So by Riesz Representation Thm $\exists$ prob. meas $\mu$ on $X$: $\tilde{B} f = \int_X f \, d\mu$ $\forall f \in C^\circ(X)$. Then $\mu_{n_\ell} \xrightarrow[\text{weakly}]{} \mu$ $\square$

# §1.4. Invariant measures

Let $X$ be a compact metric space and $T: X \to X$ be a (Borel) measurable map.

Defn. Let $\mu$ be a probability measure on $X$. We say $\mu$ is T-invariant, if

$$\mu(T^{-1}(A)) = \mu(A) \quad \forall A \subset X \text{ Borel}.$$

Equivalently, $\int_X f \circ T \, d\mu = \int_X f \, d\mu \quad \forall f \in L^1(X, \mu)$.

Remark The set of T-invariant mesures is convex: $\mu_1, \mu_2$ T-inv., $\alpha \in [0,1]$
$\Downarrow$
$\alpha\mu_1 + (1-\alpha)\mu_2$ is also T-inv.

and closed (if $T$ continuous): $\mu_n \to \mu$ weakly, $\mu_n$ T-inv. $\Rightarrow \mu$ T-inv.

Thm (Krylov-Bogolubov) Assume that $T$ is continuous. Then there exists a T-invariant probability measure.

<u>Proof</u> Fix $x_0 \in X$

and consider for $n \in \mathbb{N}$ the

<u>empirical measure</u> $\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j(x_0)}$,

i.e. $\forall f : X \to \mathbb{R}$ Borel measurable,

$$\int_X f \, d\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(x_0)) \quad \left(= \langle f \rangle_n (x_0)\right)$$

By Compactness Thm, there is a subsequence $\underline{\mu_{n_\ell}}$ which converges weakly to some prob. measure $\mu$.

We claim that $\mu$ is $T$-invariant.

Since $C^0(X)$ is dense in $L^1(X, \mu)$, enough to show that $\forall f \in C^0(X)$,

$$\int_X f \circ T \, d\mu = \int_X f \, d\mu.$$

Now, $\int_X f \circ T \, d\mu = \lim_{\ell \to \infty} \int_X f \circ T \, d\mu_{n_\ell}$

$$= \lim_{\ell \to \infty} \frac{1}{n_\ell} \sum_{j=0}^{n_\ell - 1} f(T^{j+1}(x_0)) = \lim_{\ell \to \infty} \frac{1}{n_\ell} \sum_{j=0}^{n_\ell - 1} f(T^j(x_0))$$

$$+ \lim_{\ell \to \infty} \frac{1}{n_\ell} \left( f(T^{n_\ell}(x_0)) - f(x_0) \right) = \int_X f \, d\mu. \qquad \square$$

We finally discuss

unique ergodicity:

**Thm.** Let $X$ be a compact metric space, $T: X \to X$ be continuous and $\mu_0$ be a $T$-invariant prob. measure. TFAE:

1. $\mu_0$ is the only $T$-invariant prob. meas.
2. For each $f \in C^0(X)$ and all $x \in X$ we have
$$\frac{1}{n} \sum_{j=0}^{n-1} f(T^j(x)) \xrightarrow[n\to\infty]{} \int_X f \, d\mu_0 .$$

**Proof**: see Pset 1.

**Remark** One can replace pointwise convergence by uniform convergence in 2 above. The maps satisfying 1 above are called uniquely ergodic.

**Example**: $T =$ irrational shift (see §1.1)
$\mu_0 =$ Lebesgue measure.