18.03 RECITATION SHEET WEEK 9 SOLUTIONS

(Questions with * are optional)

1. Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 \end{pmatrix}$.

(a) Find a basis for and the dimension of the null space of A.

The basis for the null space $NS(A)$ can be found with the help of the solution to the equation $Ax = 0$. We solve this by writing the RREF of this system:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & -1 & -1 & -1 & 0 \end{array}\right)$$

$R2 \to R2 + R1$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & -1 & 0 \end{array}\right)$$

$R3 \to R3 - R1$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & -2 & 0 \end{array}\right)$$

$R2 \leftrightarrow R3$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$R2 \to R2/(-2)$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Finally, we use back substitution $R1 \to R1 - R2$ to get

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

In equation form we have

$$\begin{cases} x_1 + x_2 = 0 \\ x_3 + x_4 + x_5 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = -x_2 \\ x_3 = -x_4 - x_5 \end{cases}$$

The values x_1 *and* x_3 *corresponding to the pivot columns are given by the free variables* x_2*,* x_4*, and* x_5*. From the first equation we get a basis vector*

$$\begin{pmatrix} -x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

while the second equation gives $(0, 0, -x_4 - x_5, x_4, x_5)^T =$

$$\begin{pmatrix} 0 \\ 0 \\ -x_4 - x_5 \\ x_4 \\ x_5 \end{pmatrix} = x_4 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

From the solution we can read a basis for the null space given by the vectors multiplied by x_2*,* x_4*, and* x_5*. There are three such vectors so the dimension of the null space is* $\underline{3}$*.*

(b) Find a basis for and the dimension of the column space of A.

We use the RREF form of A. The basis is given by the column vectors of A corresponding to the pivot columns. We have the row-reduced echelon form

$$\begin{pmatrix} \textcircled{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding columns of A are $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$*. There are two of these vectors so they span a vector space with the dimension* $\underline{2}$*. Note that the basis is not unique, any two non-equal linear combinations of these vectors will do.*

(c) What is the rank of A? Explain why the rank-nullity theorem holds for A.

*The rank of A is equal to the dimension of the column space i.e. it is 2. Rank-nullity theorem states that the rank of matrix A plus the nullity of A (dimension of the null space of A) equals the number of columns in A. We have shown in parts **(a)** and **(b)** that this is true. In general this is true because rank corresponds to the number of pivot columns while the nullity corresponds to the number of non-pivot columns.*

(d) Are the columns of A linearly independent? Do they span $\mathbb{R}^3$?

There are five columns in A while the column vectors are 3-dimensional. Therefore there can be at most 3 linearly independent column vectors. However, we showed that the column space is 2-dimensional (rank is 2) implying that the columns do not span $\mathbb{R}^3$.

2. Assume that a 5×3 matrix A has rank 3.

(a) What are the dimensions of the nullspace of A and the column space of A?

According to the rank-nullity theorem the dimension of the nullspace (nullity) + the rank is equal to the number of columns, 3. Therefore the nullity of A is $\underline{0}$.

(b) Are the columns of A linearly independent? Do they span $\mathbb{R}^5$?

The rank is 3 and there are 3 vectors. This means that they are linearly independent. However, they do not span $\mathbb{R}^5$. For that one would need 5 linearly independent vectors.

(c) Could it be that the equation $A\vec{x} = \vec{b}$ has no solution for some $\vec{b}$? Could it be that this equation has more than one solution for some $\vec{b}$?

It is entirely possible that $A\vec{x} = \vec{b}$ does not have a solution. Since the column vectors do not span $\mathbb{R}^5$, there are vectors $\vec{b} \in \mathbb{R}^5$ that cannot be obtained as a linear combination of the columns of A.

The non-unique solutions are expressed as $\vec{x} = \vec{x}_0 + \vec{y}$ with $\vec{y}$ in the null space of A. However, the dimension of the null space is 0, which implies that $\vec{y} = 0$. Hence any solutions are unique.

***3.** Let A be an $n \times n$ matrix and assume that the null space of A is equal to the column space of A. Show that $A^2 = 0$.

The column space is comprised of all the vectors $A\vec{x} \in \mathbb{R}^n$ s.t. $\vec{x} \in \mathbb{R}^n$. On the other hand, the null space contains all the vectors $\vec{y} \in \mathbb{R}^n$ s.t. $A\vec{y} = 0$. If these spaces are the same it follows that any $A\vec{x}$ is an element in the null space. This implies that $A(A\vec{x}) = A^2\vec{x} = 0$ for any $\vec{x} \in \mathbb{R}^n$. It follows that $A^2 = 0$.

4. For which values of the real parameter c is the matrix $A_c = \begin{pmatrix} 1 & c \\ 2c & 8 \end{pmatrix}$ invertible? Find a formula for the inverse A_c^{-1}.

We can solve for the inverse by solving the system

$$\left(\begin{array}{cc|cc} 1 & c & 1 & 0 \\ 2c & 8 & 0 & 1 \end{array}\right).$$

$R2 \to R2 - 2cR1$

$$\left(\begin{array}{cc|cc} 1 & c & 1 & 0 \\ 0 & 8-2c^2 & -2c & 1 \end{array}\right).$$

$R2 \to R2/(8-2c^2)$.

$$\left(\begin{array}{cc|cc} 1 & c & 1 & 0 \\ 0 & 1 & -2c/(8-2c^2) & 1/(8-2c^2) \end{array}\right).$$

Note that this is possible if $8-2c^2 \neq 0$ i.e. $c \neq \pm 2$. This is the condition of invertibility. We use back substitution: $R1 \to R1 - cR2$.

$$\left(\begin{array}{cc|cc} 1 & 0 & 1+2c^2/(8-2c^2) & -c/(8-2c^2) \\ 0 & 1 & -2c/(8-2c^2) & 1/(8-2c^2) \end{array}\right).$$

The inverse is given by

$$\frac{1}{8-2c^2}\begin{pmatrix} 8-2c^2+2c^2 & -c \\ -2c & 1 \end{pmatrix} = \frac{1}{8-2c^2}\begin{pmatrix} 8 & -c \\ -2c & 1 \end{pmatrix}.$$

Note that $8-2c^2 = \det(A)$ so the condition of the invertibility is $\det(A) \neq 0$.

5. Consider the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

(a) Is the matrix A invertible? If so, find the inverse.

In order to figure out if A is invertible, we calculate $\det(A)$. The value of $\det(A)$ is invariant to all the row operations except multiplying a row with a constant. We can use this to calculate the determinant. Let us swap rows $R2$ and $R3$. This gives

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now we swap $R1$ and $R2$ giving

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is an upper triangular matrix so the determinant is given by the product of the entries on the diagonal. We get $\det(A) = 1$ implying that the matrix is invertible. To find the inverse we repeat the row operations with the augmented matrix

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

$R2 \leftrightarrow R3$

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right).$$

$R1 \leftrightarrow R2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right).$$

The inverse is given by

$$A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(*b) Find the eigenvalues and eigenvectors of A

We find the eigenvalues by calculating the characteristic polynomial $p(\lambda)$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{vmatrix} = \lambda \begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} - (-1) \cdot \begin{vmatrix} 0 & -1 \\ -1 & \lambda \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & \lambda \\ -1 & 0 \end{vmatrix}.$$

Calculating the subdeterminants gives $p(\lambda) = \lambda^3 - 1$. We solve for the complex roots giving $\lambda_1 = 1$, $\lambda_2 = e^{2i\pi/3}$, and $\lambda_3 = e^{-2i\pi/3}$. We find the eigenvectors by calculating the null space of $\lambda I - A$. For λ_1 we solve

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right)$$

$R3 \to R3 + R1$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

$R3 \to R3 + R2$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$R1 \to R1 + R2$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Variable x_3 is the free variable. We get $x_1 = x_3 = x_2$. This gives an eigenvector

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We repeat the process for $\lambda_2 = e^{2i\pi/3}$. We know that the last row is a linear combination of the first two rows. Therefore the last row can be neglected.

$$\left(\begin{array}{ccc|c} e^{2i\pi/3} & -1 & 0 & 0 \\ 0 & e^{2i\pi/3} & -1 & 0 \end{array}\right)$$

We can read from this $x_2 = e^{2i\pi/3}x_1$ and $x_3 = e^{2i\pi/3}x_2$. Choosing x_1 as the free variable gives $x_2 = e^{2i\pi/3}x_1$ and $x_3 = e^{-2i\pi/3}x_1$. The solution is

$$x_1 \begin{pmatrix} 1 \\ e^{2i\pi/3} \\ e^{-2i\pi/3} \end{pmatrix} = x_1\vec{v}_2.$$

Because A is real and $\lambda_3 = \lambda_2^*$, it follows that

$$\vec{v}_3 = (\vec{v}_2)^* = \begin{pmatrix} 1 \\ e^{-2i\pi/3} \\ e^{2i\pi/3} \end{pmatrix}.$$

(*c) Diagonalize A, i.e. write it as $A = SDS^{-1}$ where D is a diagonal matrix.

We have already

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2i\pi/3} & 0 \\ 0 & 0 & e^{-2i\pi/3} \end{pmatrix}$$

and

$$S = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2i\pi/3} & e^{-2i\pi/3} \\ 1 & e^{-2i\pi/3} & e^{2i\pi/3} \end{pmatrix}.$$

We calculate the inverse of S.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & e^{2i\pi/3} & e^{-2i\pi/3} & 0 & 1 & 0 \\ 1 & e^{-2i\pi/3} & e^{2i\pi/3} & 0 & 0 & 1 \end{array}\right)$$

$R2 \to R2 - R1$ and $R3 \to R3 - R1$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & e^{2i\pi/3} - 1 & e^{-2i\pi/3} - 1 & -1 & 1 & 0 \\ 0 & e^{-2i\pi/3} - 1 & e^{2i\pi/3} - 1 & -1 & 0 & 1 \end{array}\right)$$

$R3 \to R3 + e^{-2i\pi/3}R2$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & e^{2i\pi/3} - 1 & e^{-2i\pi/3} - 1 & -1 & 1 & 0 \\ 0 & 0 & 2e^{2i\pi/3} - e^{-2i\pi/3} - 1 & -1 - e^{-2i\pi/3} & e^{-2i\pi/3} & 1 \end{array}\right)$$

We write in Cartesian form using $e^{2i\pi/3} = -1/2 + i\sqrt{3}/2$ *giving*

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3/2+i\sqrt{3}/2 & -3/2-i\sqrt{3}/2 & -1 & 1 & 0 \\ 0 & 0 & -3/2+3i\sqrt{3}/2 & -1/2+i\sqrt{3}/2 & -1/2-i\sqrt{3}/2 & 1 \end{array}\right)$$

$R2 \to [-\left(3+i\sqrt{3}\right)/6]R2$ *and* $R3 \to [-(1+i\sqrt{3})/6]R3$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \left(1+i\sqrt{3}\right)/2 & \left(3+i\sqrt{3}\right)/6 & -\left(3+i\sqrt{3}\right)/6 & 0 \\ 0 & 0 & 1 & 1/3 & \left(-1+i\sqrt{3}\right)/6 & -(1+i\sqrt{3})/6 \end{array}\right)$$

$R1 \to R1 - R2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1/2-i\sqrt{3}/2 & 1/2-i\sqrt{3}/6 & 1/2+i\sqrt{3}/6 & 0 \\ 0 & 1 & \left(1+i\sqrt{3}\right)/2 & \left(3+i\sqrt{3}\right)/6 & -\left(3+i\sqrt{3}\right)/6 & 0 \\ 0 & 0 & 1 & 1/3 & \left(-1+i\sqrt{3}\right)/6 & -(1+i\sqrt{3})/6 \end{array}\right)$$

$R1 \to R1 - (1/2 - i\sqrt{3}/2)R3$ *and* $R2 \to R2 - \left(1/2 + i\sqrt{3}/2\right)R2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & 1/3 & -\left(1+i\sqrt{3}\right)/6 & \left(-1+i\sqrt{3}\right)/6 \\ 0 & 0 & 1 & 1/3 & \left(-1+i\sqrt{3}\right)/6 & -(1+i\sqrt{3})/6 \end{array}\right)$$

The right hand side gives

$$S^{-1} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & -\left(1+i\sqrt{3}\right)/6 & \left(-1+i\sqrt{3}\right)/6 \\ 1/3 & \left(-1+i\sqrt{3}\right)/6 & -(1+i\sqrt{3})/6 \end{pmatrix}.$$

6. Consider the matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

(a) Find the eigenvectors and eigenvalues of A.

For a 2×2 *system the characteristic equation is*

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 2.$$

We solve $\underline{\lambda_1 = 1+i}$ *and* $\underline{\lambda_2 = 1-i}$. *The eigenvalues are complex conjugates of each other implying that that is the case for the eigenvectors as well. We calculate the eigenvector for* λ_1 *from the 1st equation of* $(I\lambda_1 - A)x = 0$*:*

$$(1+i-1)x_1 + x_2 = ix_1 + x_2 = 0$$

finding that $x_1 = ix_2$. *Now, the eigenvectors are*

$$\vec{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}, \quad \vec{v}_2 = (\vec{v}_1)^* = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

(b) Diagonalize A, i.e. write it as $A = SDS^{-1}$ where D is a diagonal matrix.

We have

$$D = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$$

and

$$S = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}.$$

We calculate S^{-1}*:*

$$\left(\begin{array}{cc|cc} i & -i & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right)$$

$R2 \to R2 + iR1$

$$\left(\begin{array}{cc|cc} i & -i & 1 & 0 \\ 0 & 2 & i & 1 \end{array}\right)$$

$R2 \to R2/2$ *and* $R1 \to -i * R1$

$$\left(\begin{array}{cc|cc} 1 & -1 & -i & 0 \\ 0 & 1 & i/2 & 1/2 \end{array}\right)$$

$R1 \to R1 + R2$

$$\left(\begin{array}{cc|cc} 1 & 0 & -i/2 & 1/2 \\ 0 & 1 & i/2 & 1/2 \end{array}\right)$$

We read off

$$S^{-1} = \frac{1}{2}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}$$

(c) Compute the 10th power A^{10}. (Hint: use the diagonalization. To compute the 10th power of D use the polar form of the complex eigenvalues of A.)

We have $A^{10} = SDS^{-1}SDS^{-1}...SDS^{-1} = SD^{10}S^{-1}$*. For* D^{10} *we need to calculate* λ_1^{10} *and* λ_2^{10}*. We have* $1+i = \sqrt{2}e^{i\pi/4}$*. Now,* $(\sqrt{2}e^{i\pi/4})^{10} = 32e^{i5\pi/2} = 32e^{i\pi/2} = 32i$*.* $\lambda_2^{10} = (\lambda_1^{10})^* = -32i$*. This gives*

$$A^{10} = \frac{1}{2}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1^{10} & 0 \\ 0 & \lambda_2^{10} \end{pmatrix}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} = 16i\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}.$$

Calculating the latter matrix product gives

$$A^{10} = 16i\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -i & 1 \\ -i & -1 \end{pmatrix}.$$

Finally,

$$A^{10} = 16i\begin{pmatrix} 0 & 2i \\ -2i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -32 \\ 32 & 0 \end{pmatrix}.$$

***7.** Assume that a diagonalizable $n \times n$ matrix A has only eigenvalues 1 and -1. Show that $A^2 = I$.

Since A is diagonalizable, There are matrices S comprised of eigenvectors of A and a diagonal matrix D with the eigenvalues of A s.t. $A = SDS^{-1}$. Now, $A^2 = SDS^{-1}SDS^{-1} = SD^2S^{-1}$. D^2 has the squares of the eigenvalues on the diagonal. Since the eigenvalues $\lambda_i = \pm 1$, $\lambda_i^2 = 1$. It follows that $D^2 = I$. This implies that $A^2 = SIS^{-1} = I$.