18.03 RECITATION SHEET WEEK 9

(Questions with * are optional)

1. Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 \end{pmatrix}$.

(a) Find a basis for and the dimension of the null space of A.
(b) Find a basis for and the dimension of the column space of A.
(c) What is the rank of A? Explain why the rank-nullity theorem holds for A.
(d) Are the columns of A linearly independent? Do they span $\mathbb{R}^3$?

2. Assume that a 5×3 matrix A has rank 3.

(a) What are the dimensions of the nullspace of A and the column space of A?
(b) Are the columns of A linearly independent? Do they span $\mathbb{R}^5$?
(c) Could it be that the equation $A\vec{x} = \vec{b}$ has no solution for some $\vec{b}$? Could it be that this equation has more than one solution for some $\vec{b}$?

***3.** Let A be an $n \times n$ matrix and assume that the null space of A is equal to the column space of A. Show that $A^2 = 0$.

4. For which values of the real parameter c is the matrix $A_c = \begin{pmatrix} 1 & c \\ 2c & 8 \end{pmatrix}$ invertible? Find a formula for the inverse A_c^{-1}.

5. Consider the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

(a) Is the matrix A invertible? If so, find the inverse.
(*b) Find the eigenvalues and eigenvectors of A
(*c) Diagonalize A, i.e. write it as $A = SDS^{-1}$ where D is a diagonal matrix.

6. Consider the matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

(a) Find the eigenvectors and eigenvalues of A.
(b) Diagonalize A, i.e. write it as $A = SDS^{-1}$ where D is a diagonal matrix.
(c) Compute the 10th power A^{10}. (Hint: use the diagonalization. To compute the 10th power of D use the polar form of the complex eigenvalues of A.)

***7.** Assume that a diagonalizable $n \times n$ matrix A has only eigenvalues 1 and -1. Show that $A^2 = I$.