18.03 RECITATION SHEET WEEK 8

1. Consider the system of ODEs

$$y_1' = y_1 + y_2, \qquad y_2' = y_2 - 4y_1.$$

a.) Find the general solution.
solution: First we write the system in matrix form

$$\vec{y}' = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \vec{y}$$

Next we find the eigenvalues and eigenvectors of this matrix. We apply the quadratic fomula to the equation $\lambda^2 - Tr(A)\lambda + det(A) = 0$ to get that $\lambda = 1 \pm 2i$. Next we go back and use $\lambda = 1 + 2i$ to find that the associated eigenvector is $\begin{pmatrix} 1 \\ 2i \end{pmatrix}$. This provides us with one of the complex solutions

$$\vec{y}_{cx} = e^{(1+2i)t} \begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

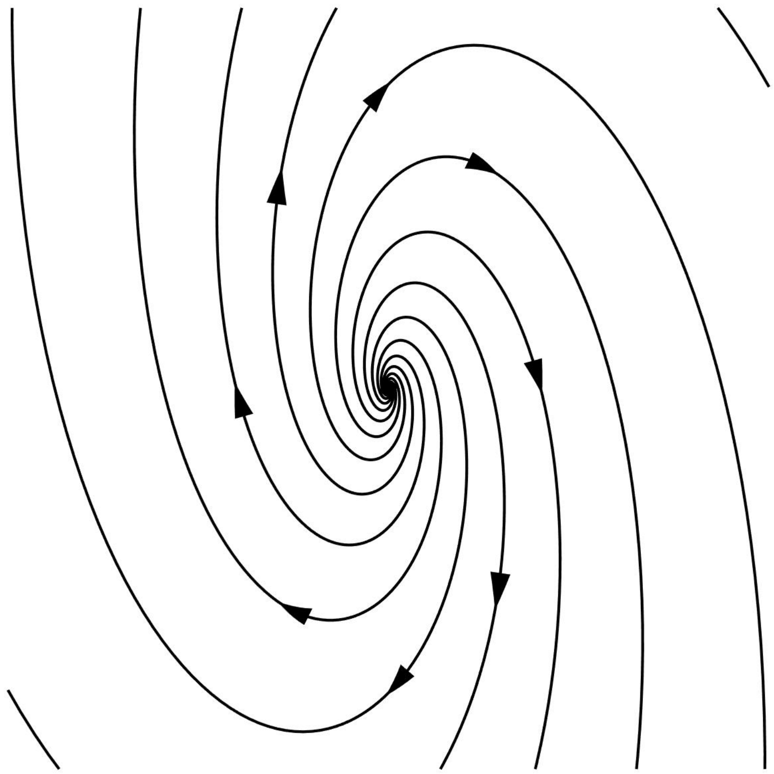
In order to get a basis for the real solution space we need to take the real and imaginary part of this solution. We start by rewriting $\vec{y}_{cx}$ so that it's easier to take the real and imaginary part.

$$\vec{y}_{cx} = e^{(1+2i)t} \begin{pmatrix} 1 \\ 2i \end{pmatrix} = \begin{pmatrix} e^t \cos(2t) + ie^t \sin(2t) \\ 2ie^t \cos(2t) - 2e^t \sin(2t) \end{pmatrix}$$

We can now read off the real and imaginary parts and use them as a basis for the particular solution.

$$\begin{aligned} \vec{y} &= C_1 Re(\vec{y}_{cx}) + C_2 Im(\vec{y}_{cx}) \\ &= C_1 \begin{pmatrix} e^t \cos(2t) \\ -2e^t \sin(2t) \end{pmatrix} + C_2 \begin{pmatrix} e^t \sin(2t) \\ 2e^t \cos(2t) \end{pmatrix} \\ &= C_1 e^t \begin{pmatrix} \cos(2t) \\ -2\sin(2t) \end{pmatrix} + C_2 e^t \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix} \end{aligned}$$

b.) Draw the phase portrait. What kind of phase portrait is it?



This is a spiral source.

c.) Draw the system on the trace-determinant plane.

The trace is 2, the determinant is 5. The point at (2,5) is above the $det = Tr^2/4$ parabola and Tr is positive.

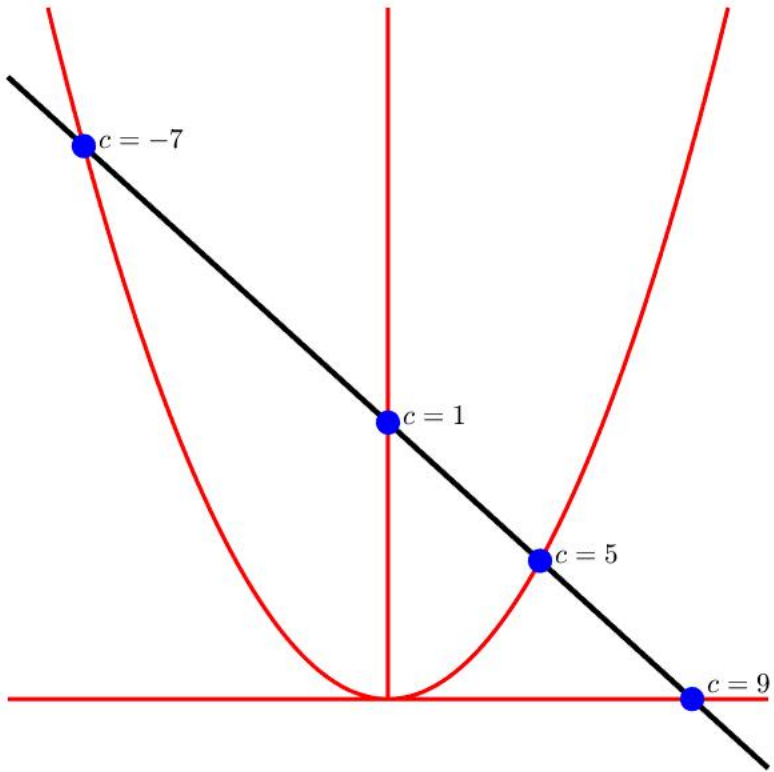
2. The romantic dynamics of Romeo and Juliet is modeled by the system of ODEs

$$R' = -R + 3J, \quad J' = -3R + cJ.$$

Here c is an arbitrary real parameter and $R(t), J(t)$ are the affection of Romeo for Juliet and the affection of Juliet for Romeo at time t. (Romeo's affection for Juliet, whether positive or negative, tends to decline by itself. However, it is also positively influenced by Juliet's affection for Romeo. On the other hand, the more Romeo likes Juliet, the more she tends towards disliking him.)

a.) Plot this system of ODEs on the trace-determinant plane for all choices of c. Mark the intersection of the resulting set of points with the critical parabola.

solution: The trace is $c - 1$ and the determinant is $9 - c$. If we are on the critical parabola, then $4det = tr^2$, from this we get a quadratic equation in c: $c^2 + 2c - 35 = 0$, whose solutions are the values of c where we are on the critical parabola. The solutions of this quadratic are $c = -7$ and $c = 5$. The corresponding tr-det points are $(-8, 16)$ and $(4, 4)$.



b.) What kind of phase portrait does the system have? Your answer will depend on c. You may skip the cases of double eigenvalue or when one of the eigenvalues equals 0, since we did not cover these portraits in class.

solution: Using the picture from part (a) and the description of what the different regions of the Tr-Det plane mean from lecture we learn that:

- For $c < -7$, it's a sink node.
- For $-7 < c < 1$, it's a spiral sink.
- For $c = 1$, it's a center.
- For $1 < c < 5$, it's a spiral source.
- For $5 < c < 9$, it's a source node.
- For $c > 9$, it's a saddle.

Note that tr $= 0$ corresponds to $c = 1$ and det $= 0$ corresponds to $c = 9$.

c.) For which values of c is the system stable? For which values of c is it semistable?

solution From the lecture notes we know how to read off stability from the tr-det plane. It's stable in the second quadrant and semistable on the boundaries of the second quadrant (the origin is special are requires more info). From this we learn that the system is stable for $c < 1$, semistable for $c = 1$ and unstable for $c > 1$.

3. Bring the following systems of linear equations to reduced row echelon form and find the general solution to each of them.

a.)

$$-3x_1 + x_2 = -1,$$
$$2x_1 + x_2 = 4,$$
$$x_2 - x_1 = 1.$$

solution: We begin by setting up the associated augmented matrix.

$$\left(\begin{array}{cc|c} -3 & 1 & -1 \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{array}\right)$$

Next we use row operations to bring it into REF. $R2 \to R2 + (2/3)R1$

$$\left(\begin{array}{cc|c} -3 & 1 & -1 \\ 0 & 5/3 & 10/3 \\ -1 & 1 & 1 \end{array}\right)$$

$R3 \to R3 - (1/3)R1$

$$\left(\begin{array}{cc|c} -3 & 1 & -1 \\ 0 & 5/3 & 10/3 \\ 0 & 2/3 & 4/3 \end{array}\right)$$

$R3 \to R3 - (2/5)R2$

$$\left(\begin{array}{cc|c} -3 & 1 & -1 \\ 0 & 5/3 & 10/3 \\ 0 & 0 & 0 \end{array}\right)$$

Next we go from REF to RREF. $R1 \to (-1/3)R1$, $R2 \to (3/5)R2$,

$$\left(\begin{array}{cc|c} 1 & -1/3 & 1/3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

$R1 \to R1 + (1/3)R2$

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

This final matrix is the Reduced row echelon form. From it we can read off the general solution: $x_1 = 1$, $x_2 = 2$

b.)

$$x_1 + x_2 + x_3 + x_4 = 4,$$

$$x_1 + x_2 - x_3 - x_4 = 2.$$

solution: We begin by setting up the associated augmented matrix.

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 1 & -1 & -1 & 2 \end{array}\right)$$

Next we use row operations to bring it into REF. $R2 \to R2 - R1$,

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 0 & -2 & -2 & -2 \end{array}\right)$$

$R2 \to (-1/2)R2$,

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

Next we go from REF to RREF. $R1 \to R1 - R1$,

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

This final matrix is the Reduced row echelon form. From it we can read off the general solution: $x_1 = 3 - x_2$, $x_3 = 1 - x_4$ and x_2, x_4 can be any numbers.

c.)

$$x_1 + 2x_2 + x_3 = 4,$$

$$x_1 + 2x_2 + 2x_3 = c,$$

$$x_1 + 2x_2 = 3,$$

where c is any number (your answer will depend on c).

solution: We begin by setting up the associated augmented matrix.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 1 & 2 & 2 & c \\ 1 & 2 & 0 & 3 \end{array}\right)$$

Next we use row operations to bring it into REF. $R2 \to R2 - R1$ and $R3 \to R3 - R1$,

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & c-4 \\ 0 & 0 & -1 & -1 \end{array}\right)$$

$R3 \to R3 + R2$,

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & c-4 \\ 0 & 0 & 0 & c-5 \end{array}\right)$$

Next we go from REF to RREF. $R1 \to R1 - R2$,

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 8-c \\ 0 & 0 & 1 & c-4 \\ 0 & 0 & 0 & c-5 \end{array}\right)$$

This final matrix is the Reduced row echelon form. From it we can read off the general solution: If $c = 5$, then $x_1 = 3 - 2x_2$, $x_3 = 1$ and x_2 can be any number. If $c \neq 5$, there is no solution.