18.03 RECITATION SHEET WEEK 7 – SOLUTIONS

1. Consider the ODE $y'' + y' - 2y = 0$.

(a) Write the corresponding companion system in matrix form (which will be a 2×2 first order system of ODEs).
Solution:
We introduce the new variables $y_1 = y$ and $y_2 = y'$. With these, the companion system is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= 2y_1 - y_2. \end{aligned}$$

In matrix form, we obtain

$$\vec{y}' = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \vec{y},$$

where $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

(b) Find the general solution of the companion system using the method of Section 6.2 in the lecture notes.
Solution:
We proceed by first computing the eigenvalues of the companion matrix:

$$\begin{aligned} \lambda_{1,2} &= \frac{\mathrm{tr}(A)}{2} \pm \sqrt{\frac{\mathrm{tr}(A)^2}{4} - \det(A)} \\ &= -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2} \\ &= -2, 1. \end{aligned}$$

Next, we find the eigenvector to eigenvalue $\lambda_1 = -2$ by solving the linear system

$$\begin{aligned} &\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \vec{x} = -2\vec{x} \\ \Leftrightarrow &\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \vec{x} = 0. \end{aligned}$$

The space of solutions is given by the vectors with $x_2 = -2x_1$, so it is spanned by the eigenvector

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

We now find the eigenvector to eigenvalue $\lambda_2 = 1$ by solving the linear system

$$\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \vec{x} = \vec{x}$$
$$\Leftrightarrow \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \vec{x} = 0.$$

The space of solutions is given by the vectors with $x_2 = x_1$, so it is spanned by the eigenvector

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

With this, the general solution can be written in terms of eigenvalues and eigenvectors as

$$\begin{aligned} \vec{y}(t) &= C_1 e^{-2t} \vec{v}_1 + C_2 e^t \vec{v}_2 \\ &= C_1 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

(c) Find the solutions of the system satisfying the following initial conditions: *(i)* $y(0) = 1$, $y'(0) = 1$, *(ii)* $y(0) = 6$, $y'(0) = 0$.
Solution:

Let's compute

$$\vec{y}(0) = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 + C_2 \\ -2C_1 + C_2 \end{pmatrix}$$

For part (i) we read off the equations $C_1 + C_2 = 1$ and $-2C_1 + C_2 = 1$, which is solved by $C_1 = 0$ and $C_2 = 1$. Thus, the solution is

$$\vec{y}(t) = e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For part (ii), the equations are $C_1 + C_2 = 6$ and $-2C_1 + C_2 = 0$ which is solved by $C_1 = 2$ and $C_2 = 4$. Thus, the solution is

$$\vec{y}(t) = 2e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + 4e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(d) Plot the trajectories of the solutions from part (c)
Solution:

(i) The solution we found was

$$\vec{y}(t) = e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so for $t \to -\infty$ it starts out at the origin and then remains directed along the ray given by the vector $(1,1)$. It is drawn in the picture below in red.

(ii) The solution from this part was

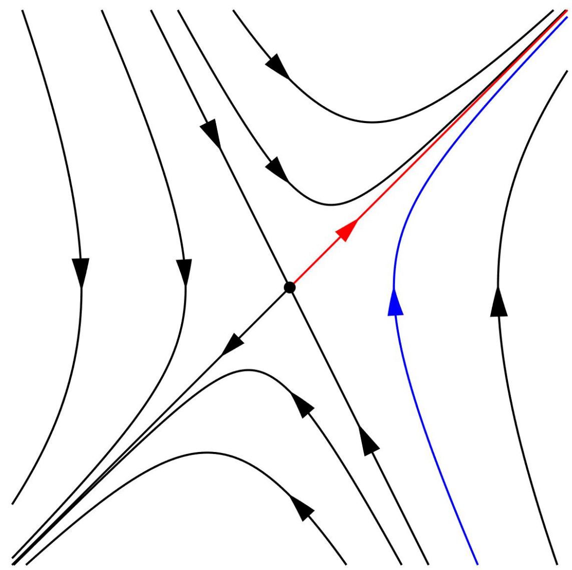
$$\vec{y}(t) = 2e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + 4e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

To draw this, we look at the limits again. For $t \to -\infty$, the term proportional to e^t vanishes, so the trajectory is parallel to the vector $(1,-2)$. For $t \to \infty$, the term proportional to e^{-2t} vanishes, and the trajectory is parallel to $(1,1)$. At $t = 0$, the trajectory is at the point $(6,0)$. The trajectory is shown in the picture below in blue.

(e) Draw the phase portrait of the system. What type is it? Be sure to include all the special trajectories (the ones which are straight rays).
Solution:

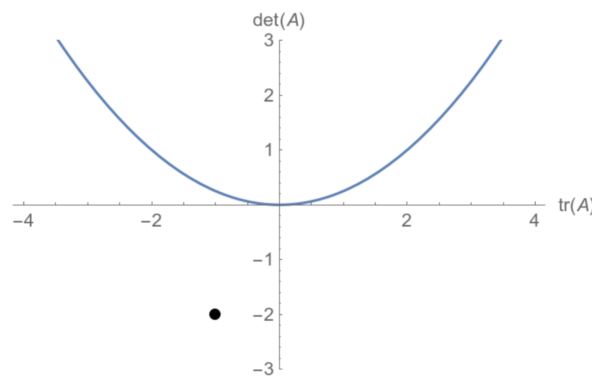
We can sketch the phase portrait by starting with the special trajectories, which are straight rays in the directions of the eigenvectors of A. The trajectories in the direction of eigenvectors corresponding to positive eigenvalues are directed away from the origin (exponential growth), and those in the direction of eigenvectors corresponding to negative eigenvalues are directed towards the origin (exponential decay).

Since we have one positive and one negative eigenvalue, this phase portrait is a *saddle*, and all other trajectories start our tangent to the rays corresponding to negative eigenvalues, and approach the rays tangent to the positive eigenvalues:



The red ray corresponds to the trajectory from part (d) (i), the blue trajectory is the one from part (d) (ii).

(f) Plot the companion system in the trace-determinant plane.
Solution:
The trace of the companion matrix is $\mathrm{tr}(A) = -1$ and the determinant is $\det(A) = -2$. We plot them in the trace-determinant plane:



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The curve $\det(A) = \frac{1}{4}\,\mathrm{tr}(A)^2$ is shown in blue.

2. Consider a system of 2 connected tanks with one of them leaking (see Section 6.1 of the lecture notes) which is described by the ODEs $y_1' = y_2 - y_1$, $y_2' = y_1 - 2y_2$.

(a) Write the system in vector form.
Solution:

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \vec{y}\,' = A\vec{y}, \quad A = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}.$$

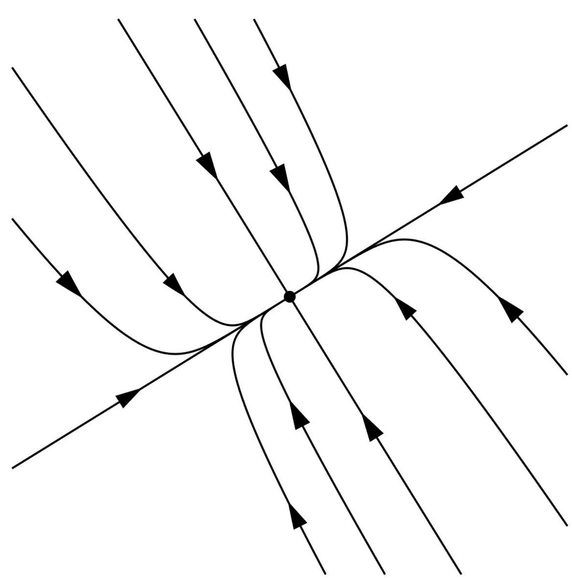
(b) Find the general solution to the system.
Solution:
As before we first find the eigenvalues: $\lambda_1 = \frac{-3-\sqrt{5}}{2}$, $\lambda_2 = \frac{-3+\sqrt{5}}{2}$, The we solve the linear systems $A\vec{x} = \lambda_{1,2}\vec{x}$ to obtain the eigenvectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -\frac{1+\sqrt{5}}{2} \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{pmatrix}.$$

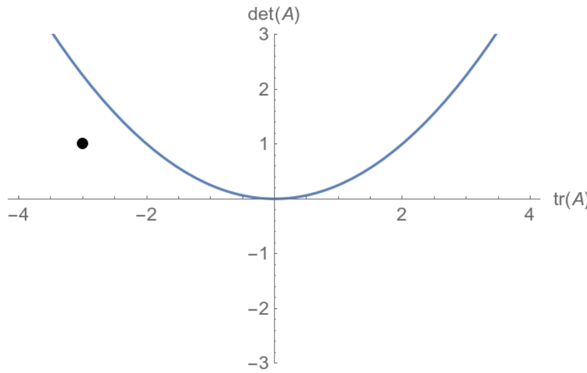
(c) Draw the phase portrait of the system. What type is it? Be sure to include all the special trajectories (the ones which are straight rays).

The companion matrix has two negative eigenvalues, so this system is a sink node. Again, the special straight ray trajectories are along the directions of the two eigenvectors, but here, both are directed towards the origin (exponential decay).

All other trajectories start out parallel to the direction corresponding to the smaller eigenvalue, and then curve parallel to the direction of the larger eigenvalue, until they finally reach the origin.



(d) Plot the system in the trace-determinant plane.
Solution:
The trace of the companion matrix is $\text{tr}(A) = -2$ and the determinant is $\det(A) = 1$. We plot them in the trace-determinant plane:



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The curve $\det(A) = \frac{1}{4}\operatorname{tr}(A)^2$ is shown in blue.