1. Consider the initial value problem \( y'' + 4y = \cos(\omega t) \), \( y(0) = 0 \), \( y'(0) = 0 \) where \( \omega > 0 \) is a real parameter.

(a) Find the solution of this initial value problem when \( \omega \neq 2 \).

**Solution.** To get the general solution to the non-homogeneous ODE
\[
y'' + 4y = \cos(\omega t),
\]
we first find the solution to the homogeneous part \( y'' + 4y = 0 \). The characteristic polynomial is \( P(z) = z^2 + 4 \) with a pair of complex roots \( \pm 2i \). Therefore, the solution to the homogeneous equation \( y'' + 4y = 0 \) is
\[
y_h(t) = C_1 \cos(2t) + C_2 \sin(2t).
\]

To get a particular solution, we use complex replacement together with ERF. The right-hand side \( \cos(\omega t) \) in its complex form is \( \Re(e^{i\omega t}) \). ERF applied to the complex version of the ODE gives
\[
y_c(t) = e^{i\omega t} \frac{P(i\omega)}{(i\omega)^2 + 4} = \frac{e^{i\omega t}}{4 - \omega^2},
\]
for \( \omega \neq 2 \). Taking the real part of \( y_c \), we get a particular solution to the ODE \( 1) \),
\[
y_p(t) = \Re(y_c(t)) = \frac{\cos \omega t}{4 - \omega^2},
\]

By superposition principle, the general solution to \( 1) \) is
\[
y(t) = y_p(t) + y_h(t) = \frac{\cos \omega t}{4 - \omega^2} + C_1 \cos(2t) + C_2 \sin(2t).
\]

Plugging in the initial conditions, we get
\[
0 = y(0) = \frac{1}{4 - \omega^2} + C_1,
\]
\[
0 = y'(0) = 2C_2.
\]

Therefore, \( C_1 = \frac{1}{\omega^2 - 4} \) and \( C_2 = 0 \). Hence, the solution to the IVP is
\[
y(t) = \frac{\cos \omega t}{4 - \omega^2} - \frac{1}{4 - \omega^2} \cos(2t).
\]

\[ \square \]
(b) Find the complex gain, amplitude gain, and phase lag as functions of $\omega$ when $\omega \neq 2$.

**Solution.** Complex gain is given by

$$\text{complex gain} = \frac{1}{P(i\omega)} = \frac{1}{4 - \omega^2}.$$  

Notice that the complex gain can also be written as $re^{i\theta}$ where $r$ is the amplitude gain and $-\theta$ is the phase lag. Writing the complex gain in the polar form, we get

$$\text{complex gain} = \begin{cases} \frac{1}{|4 - \omega^2|}e^{i\theta}, & \text{if } 0 < \omega < 2, \\ \frac{1}{|4 - \omega^2|}e^{i\pi}, & \text{if } \omega > 2. \end{cases}$$

Therefore, the amplitude gain in this case is

$$\text{amplitude gain} = \frac{1}{|4 - \omega^2|},$$

and the phase lag is

$$\text{phase lag} = \begin{cases} 0, & \text{if } 0 < \omega < 2, \\ -\pi, & \text{if } \omega > 2. \end{cases}$$

\(\square\)

(c) Find the solution of this initial value problem when $\omega = 2$.

**Solution.** The solution to the homogeneous part of the ODE remains the same:

$$y_h(t) = C_1 \cos(2t) + C_2 \sin(2t).$$

To get a particular solution, we use complex replacement with ERF'. The right-hand side $\cos(2t)$ in its complex form is $Re(e^{i2t})$. (ERF no longer works now as $P(2i) = 0$.) We use ERF’ and get

$$y_p(t) = \frac{te^{i2t}}{P'(2i)} = \frac{te^{i2t}}{4i} = \frac{te^{i2t}}{4e^{i\pi/2}} = \frac{t}{4} e^{i(2t - \pi/2)}.$$  

Taking its real part, we get

$$y_p(t) = Re(y_e(t)) = \frac{t}{4} \cos(2t - \frac{\pi}{2}).$$

By superposition principle, the general solution is

$$y(t) = y_p(t) + y_h(t) = \frac{t}{4} \cos(2t - \frac{\pi}{2}) + C_1 \cos(2t) + C_2 \sin(2t).$$
Plugging in the initial conditions, we have
\[
0 = y(0) = C_1
\]
\[
0 = y'(0) = 2C_2.
\]

Therefore, the solution to the IVP is
\[
y(t) = \frac{t}{4} \cos(2t - \frac{\pi}{2}) = \frac{t}{4} \sin(2t)
\]
\[
\square
\]

(d) (*) Show that as \( \omega \to 2 \), the solution in part (a) converges to the solution in part (b) for any fixed \( t \).

Solution. For any fixed \( t \), we take the limit
\[
\lim_{\omega \to 2} \cos \omega t = \frac{1}{2} \cos(2t) = \lim_{\omega \to 2} \frac{\cos \omega t - \cos 2t}{-2\omega} = \lim_{\omega \to 2} \frac{-t \sin \omega t}{-2\omega} = \frac{\sin 2t}{4},
\]
which is precisely the solution when \( \omega = 2 \). Notice that in the second-to-last equality, we used L'Hôpital's rule and also note that since the variable in the limit is \( \omega \), the derivative should be with respect to \( \omega \).
\[
\square
\]

(e) For your entertainment, the graphs of \( y(t) \) for several values of \( \omega \) are included on the next page.

Solution. Nothing to be done here—it is for your entertainment.
\[
\square
\]

2. Which of the following ODEs are stable? For those that are not, give an example of a solution that does not go to 0 as \( t \to \infty \).

(a) \( y'' + 7y' + 8y = 0 \)

Solution. Recall that the stability test for 2nd order equation says the equation
\[
a_2y'' + a_1y' + a_0y = 0
\]
is stable if and only if \( a_0 > 0, a_1 > 0 \).

In this case, \( a_1 = 7, a_0 = 8 \). Therefore, it is \boxed{\text{stable}}.

Alternatively, one may also tell this ODE is stable by looking at the real part of the characteristic roots. Recall that an ODE \( P(D)y = 0 \) is stable if and only if the real part of every characteristic root is negative.

In this case, the roots are
\[
\frac{-7 \pm \sqrt{7^2 - 4 \cdot 8}}{2} = \frac{-7 \pm \sqrt{17}}{2}.
\]
Both of them are negative, yielding the same answer as before.
\[
\square
\]

(b) \( y'' + y' - 2y = 0 \)
Solution. By the stability test for 2nd order equations, we conclude that the equation is not stable.

Alternatively, we may also look at the characteristic roots. The characteristic roots are
\[\frac{-1 \pm \sqrt{1 + 4 \cdot 2}}{2} = \frac{-1 \pm 3}{2} = 1, -2.\]

Thus, it is not stable.

The following is a solution that does not go to 0 as \(t \to \infty\):
\[y(t) = e^t\]

\[\square\]

3. Compute the linear combination \(2\vec{a} - 3\vec{b}\) where
\[\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\]

Solution.
\[2\vec{a} - 3\vec{b} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}.\]

\[\square\]

4. Are the following sets of vectors linearly independent? For those which are linearly dependent, find a nontrivial linear combination which gives the zero vector. (Hint: write the equation that a linear combination of those is equal to 0 and solve for the coefficients.)

(a) \(\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\)

Solution. As there is a zero vector, this set is linearly dependent. A nontrivial linear combination that is equal to the zero vector is
\[0\vec{a} + 1\vec{b} = \vec{0}.\]

\[\square\]

(b) \(\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\)

Solution. To tell whether there is a nontrivial linear combination that leads to the zero vector, we solve the linear system
\[\begin{pmatrix} 0 \\ 0 \end{pmatrix} = x\vec{a} + y\vec{b} = x\begin{pmatrix} 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} x + y \\ x - y \end{pmatrix}.\]
Observe that $x = 0, y = 0$ is the **only** solution to this linear system. That means, no non-trivial linear combination is equal to the zero vector. Hence, this set is **linearly independent**.

Alternatively, **two** non-zero vectors are linearly dependent only if one is a multiple of another. In this case, this is apparently not true. Hence, they are linearly independent. □

(c) $\vec{a} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$

**Solution.** To tell whether there is a nontrivial linear combination that leads to the zero vector, we solve the linear system

$$
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = x\vec{a} + y\vec{b} + z\vec{c} = \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ y \end{pmatrix} + \begin{pmatrix} 2z \\ -z \\ -3z \end{pmatrix} = \begin{pmatrix} x + 2z \\ x + y - z \\ y - 3z \end{pmatrix}.
$$

A nonzero solution to this linear system is $x = -2, y = 3, z = 1$. Therefore, the following nontrivial linear combination is equal to the zero vector:

$$-2\vec{a} + 3\vec{b} + \vec{c} = 0.$$

Thus, this set is **linearly dependent**. □