1. Find the general solutions to the following ODEs:

(a) $ty' - y = t$, when $t > 0$
(b) $y' - (\tan t)y = 1$, when $-\pi/2 < t < \pi/2$

Solution:

We first find all $y_h$ that solve the homogeneous ODE. We use variation of parameters to find all solutions to the inhomogeneous equation.

(a) General solution to the homogeneous problem $y'_h - y_h/t = 0$ is given by $y_h = Ce^{-A(t)}$, where

$$A(t) = \int dt \left( -\frac{1}{t} \right) = -\ln |t|,$$

and $C$ is any real number. Note that we omitted the integration constant because it is taken care of by the constant $C$. Now, $y_h = Ct$ (notice that $t > 0$).

We then find all solutions to the inhomogeneous equation by substituting the constant $C$ above by a function $u(t)$, i.e. $y = u(t)t$. This gives

$$t(u't + u) - ut = t$$

This gives $t^2u' = t$ or $u' = \frac{1}{t}$. We can solve this for $u$ giving

$$u = \int dt \frac{1}{t} = \ln t + C_0.$$ 

Therefore the solutions to the inhomogeneous equation are given by

$$y = ut = t \ln t + C_0 t.$$

(b) Homogeneous problem is solved by $y_h = Ce^{-A(t)}$.

$$A(t) = \int dt (-\tan t) = \ln (\cos t).$$

Note that with the given limits $\cos t > 0$. Now, $y_h = C/\cos t$.

We then find the solution to the inhomogeneous solution $y = u(t)\frac{1}{\cos t}$. As in (a), we get $u'\frac{1}{\cos t} = 1$

$$u = \int dt \cos t = \sin t + C_0.$$
So the final solution becomes
\[
y = u(t) \frac{1}{\cos t} = \tan t + \frac{C_0}{\cos t}.
\]

2. Salting the leaky tank

A tank initially has 100 L of fresh water (i.e. no salt). A pipe delivers brine with a salt concentration of 75 g/L at a rate 1 L/min. At the same time the tank leaks at a rate of 2 L/min. The tank is constantly mixed making the concentration of salt same everywhere in the tank.

(a) Write the ODE initial value problem modeling the amount of salt in the tank at time \(t\) (in grams)

(b) Find the solution (variation of parameters)

(c) We want to stop the process (pour into a non-leaky tank) once the concentration reaches 30 g/L. When will it be and how much water will be left?

(d) The person managing the tank forgot to terminate the mixing process and all the water poured out. What was the salt concentration just before the tank became empty?

Solution:

We assume that the density of water does not change as a function of salt concentration. The attached figure explains the notation used here. For parameters we have \(w(0) := w_0 = 100, \ f_s^i = 75, \ f_o = 2, \ f_i = 1.\)

(a) We write the initial value problem for salt as
\[
y' = f_s^i - f_o^s = 75 - f_o^s, \quad y(0) = 0.
\]

The outgoing salt flow can be expressed as the salt concentration times the leaking water flow giving
\[
f_o^s = \frac{y}{w} f_o.
\]

We can solve the amount of water from
\[
w' = f_i - f_o = -1, \quad w(0) = w_0 = 100.
\]
Integrating gives \( w(t) = 100 - t \) with the given initial value. Now the DE for salt becomes

\[
y' = 75 - \frac{2}{100 - t}y, \quad y(0) = 0.
\]

(b) The equation above can be rewritten as

\[
y' + \frac{2}{100 - t}y = 75.
\]

First we solve the homogeneous equation as in Problem 1:

\[
A(t) = \int dt \frac{2}{100 - t} = -2 \ln(100 - t).
\]

The homogeneous solution is given as

\[
y_h = Ce^{-A(t)} = C (100 - t)^2.
\]

We look for the general solution using variation of parameters by replacing the constant \( C \) by a function \( u \) giving \( y = u(100 - t)^2 \). Inserting in the DE for salt gives \((100 - t)^2u' = 75\) since the zeroth order term cancels as in Problem 1. We write this as

\[
u' = 75(100 - t)^{-2}.
\]

Integrating and multiplying by the homogeneous solution \((100 - t)^2\) gives the final solution

\[
y = (100 - t)^2 \int dt \left[ 75 (100 - t)^{-2} \right] = 75 (100 - t) + C_0 (100 - t)^2.
\]

Using the initial condition \( y(0) = 0 \) gives \( C_0 = -\frac{3}{4} \). Inserting this gives

\[
y = 75 (100 - t) - \frac{3}{4} (100 - t)^2 = \frac{3}{4} t (100 - t).
\]

(c) The concentration is given by the ratio between salt and water \( y/w \). Let us solve

\[
y/w = \frac{3}{4} t (100 - t)/(100 - t) = \frac{3}{4} t = 30.
\]

Solving this gives \( t = 40 \) i.e. the concentration is 30 g/L after 40 minutes. The amount of water left is \( w(40) = 100 - 40 = 60 \) liters.

(d) The water has poured out after 100 minutes. The concentration at that time is

\[
y(100)/w(100) = 75
\]

3. Express as \( x + iy \):

\[\text{Solution:}\]
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(a) 
\[(1 + \sqrt{3}i)(a + bi) = (a - b\sqrt{3}) + i(a\sqrt{3} + b).\]

(b) 
\[
\frac{1 + \sqrt{3}i}{1 - \sqrt{3}i} = \frac{(1 + \sqrt{3}i)(1 + \sqrt{3}i)}{(1 - \sqrt{3}i)(1 + \sqrt{3}i)} = \frac{1}{1 + 3} \left((1 - 3) + 2i\sqrt{3}\right) = \frac{-1}{2} + \frac{i}{2}\sqrt{3}.
\]

(c) \[e^{i\pi/3} + e^{i2\pi/3} = \cos(\pi/3) + i\sin(\pi/3) + \cos(2\pi/3) + i\sin(2\pi/3).\] We expanded the exponential using Euler’s exponential formula. We note that \[\cos(\pi/3) = -\cos(2\pi/3) = \frac{1}{2}\] and \[\sin(\pi/3) = \sin(2\pi/3) = \frac{\sqrt{3}}{2}\] giving \[2i\sin(\pi/3) = i\sqrt{3}.
\]

4. Quadratic equations

(a) Solve the equation \[z^2 + 2z + 2 = 0.\]

Solution:
We use the quadratic formula giving \[z = -1 \pm \sqrt{-1} = -1 \pm i.\]

(*b) Show that the functions \[y(t) = e^{z_1t}\] and \[f(t) = e^{z_2t}\] where \[z_1, z_2\] are the roots of the above equation, solve the differential equation \[y'' + 2y' + 2y = 0.\]

Solution:
Inserting \[e^{zt}\] in the DE gives \[(z^2 + 2z + 2) e^{zt} = 0,\]
which is true if \[z^2 + 2z + 2 = 0.\] This is solved by \[z_1\] and \[z_2.\]

(*c) Take the real and imaginary parts of those functions to construct two real valued solutions to the equation.

Solution:
The real part \[\text{Re}(e^{zt}) = e^{-t}\cos(\pm t) = e^{-t}\cos(t)\]
and the imaginary part \[\text{Im}(e^{zt}) = e^{-t}\sin(\pm t) = \pm e^{-t}\sin(t)\]
both solve the DE. Also their linear combinations solve the same DE.
5. Complex exponential

Check by hand (differentiating real and imaginary parts) that the complex valued function \( f(t) = e^{(1+2i)t} \) solves the equation \( f' = (1 + 2i)f \).

**Solution:**

We use Euler’s formula giving

\[
f(t) = e^t e^{2it} = e^t \cos(2t) + ie^t \sin(2t).
\]

Now,

\[
f'(t) = e^t (\cos(2t) - 2\sin(2t)) + ie^t (\sin(2t) + 2\cos(2t)) = (1 + 2i)f.
\]

6. Roots: Find all complex solutions to the equations

(a) \( z^2 = i \)
(b) \( z^4 = 16 \)

**Solution:**

These are solved conveniently by writing the complex number in polar form \( z = re^{it} \). The polar representation is unique up to modulo \( 2\pi \) for the phase \( t \) meaning that if \( r_1e^{it_1} = r_2e^{it_2} \), it follows that \( r_1 = r_2 \) and \( t_1 = t_2 + 2\pi n \), where \( n \) is any integer. Using this we get

(a) \( z^2 = r^2 e^{2it} = i = 1e^{i\pi/2} \). Now we have \( r^2 = 1 \) giving \( r = 1 \) (remember that \( r \geq 0 \)). For the phase we have \( 2t = \pi/2 + 2\pi n \) implying that \( t = \pi/4 + \pi n \). For the value \( e^{it} \) it only matters if \( n \) is even or odd. Also, note that \( e^{i(t+\pi)} = -e^{it} \). Now we have

\[
z = \pm e^{i\pi/4} = \pm \frac{1+i}{\sqrt{2}}.
\]

(b) Similarly, \( r^4 e^{4it} = 16 e^{i0} \) giving \( r = \sqrt[4]{16} = 2 \). For the phase we have \( 4t = \pi/2 + 2\pi n \). Solving for \( t \) gives \( t = n\pi/2 \). Now we have 4 independent solutions for \( e^{it} \) corresponding to values \( n = 0, 1, 2, 3 \) (or any other successive four integers) giving

\[
z \in \{2e^{i0}, 2e^{i\pi/2}, 2e^{i\pi}, 2e^{3i\pi/2}\} = \{2, 2i, -2, -2i\}.
\]

*7. Integrating

We will use complex numbers to compute the antiderivative of \( e^t \sin t \).

(a) Write \( e^t \sin t = \text{Im}(e^{(1+i)t}) \)
(b) Integrate \( e^{(1+i)t} \) using the same formula as for the real exponential
(c) Take the imaginary part of the result to get the antiderivative.
Solution:

We integrate

\[ \int dt e^{(1+i)t} = \frac{1}{1+i} e^{(1+i)t} = \frac{1-i}{2} e^{(1+i)t} = \frac{e^t}{2} (1-i) e^{it}. \]

Taking the imaginary part gives the answer

\[ \frac{e^t}{2} (\sin t - \cos t). \]