18.03 RECITATION SHEET WEEK 1 SOLUTIONS

(Questions with * are optional)

1. We substitute $y(t)$ into the ODE and see if the equation holds. If there is an initial condition, then we also need to check if $y(0)$ matches this number.

(a) $y(t) = 68 + e^{-t}$  \hspace{5pt} y' = 68 - y, \hspace{5pt} y(0) = 69
Yes: $y' = -e^{-t} = 68 - y$, \hspace{5pt} y(0) = 68 + e^0 = 69.

(b) $y(t) = C_1 \cos(t) + C_2 \sin(t)$  \hspace{5pt} y'' + y = 0
Yes: $y'' + y = (-C_1 \cos(t) - C_2 \sin(t)) + (C_1 \cos(t) + C_2 \sin(t)) = 0.$

(c) $y(t) = Ce^{-t^2}$  \hspace{5pt} y' = -ty
No: $y' = Ce^{-t^2}(-2t) \neq (-t)Ce^{-t^2} = -ty$. The correct solution is $y(t) = Ce^{-t^2/2}$.

(d) $y = 1/(C - t)$  \hspace{5pt} y' = y^2
Yes: $y' = \frac{1}{(C - t)^2} = y^2$.

2. (Taxonomy)

(a) $y''' + y^2 = 2$
3rd order, nonlinear

(b) $y'' + ty = 1$
2nd order, linear, inhomogeneous, variable coefficient

(c) $y'' + 5y' + y = t$
2nd order, linear, inhomogeneous, constant coefficient (t on the right hand side does not count as a coefficient!)

(d) $y'' + y' + y/t = 0$
2nd order, linear, homogeneous, variable coefficient

*3. (Superposition Principle)

(a) $y'' + ty = 0$
Yes it satisfies Superposition Principle. Suppose $y_1(t)$ and $y_2(t)$ both satisfy
the equation, i.e. \( y_1'' + ty_1 = 0, \ y_2'' + ty_2 = 0 \), then their sum \( y_3 = y_1 + y_2 \) satisfies
\[
y_3'' + ty_3 = (y_1 + y_2)'' + t(y_1 + y_2) = (y_1'' + ty_1) + (y_2'' + ty_2) = 0.
\]

(b) \( y' = y^2 \)
No it does not satisfy the Superposition Principle. Recall from 1(d) \( y_1 = 1/(1-t) \) and \( y_2 = 1/(1-t) \) are both solutions to the ODE. However if we take the sum \( y_3 = y_1 + y_2 = 2/(1-t) \), then we can check \( y_3'' + ty_3 \neq 4/(1-t)^2 = y_3'' \).

4. We use separation of variables to solve the ODEs:

(a) \( y' = te^y \) (find the general solution)
Solution: rewrite the equation as \( \frac{dy}{e^y} = tdt \), integrate we get \(-e^{-y} = t^2/2 + C\), \( C \) is any constant. Solve out \( y \) we get
\[
y(t) = -\ln(-t^2/2 - C).
\]
(There is some subtlety with the choice of \( C \) because the function \( \ln x \) is only defined for positive \( x \).) Do not forget to check that we did not leave out constant solutions by setting \( te^{y_0} = 0 \) and see that no constant \( y_0 \) satisfies this equation.

(b) \( y' = (\sin t)y, \ y(0) = 1 \) (find the solution to the initial value problem)
Solution: we first solve the ODE by rewriting as \( \frac{dy}{y} = \sin tdt \), integrate to get \( |y| = -\cos t + C \), exponentiate we get \( |y| = e^C e^{-\cos t} \), remove absolute value to get \( y = \pm e^C e^{-\cos t} \) that is \( y = \tilde{C} e^{-\cos t}, \tilde{C} \neq 0 \). In addition find the constant solutions for \( (\sin t)y_0 = 0 \) i.e. \( y_0 = 0 \). Combining above, the final solution is
\[
y(t) = \tilde{C} e^{-\cos t}, \tilde{C} \text{ is any constant}.
\]
(Alternatively we can use the formula for the solution of a first order linear homogeneous equation from §2.2.1: the equation is \( y' + a(t)y = 0 \) where \( a(t) = -\sin t \), an antiderivative of \( a(t) \) is given by \( A(t) = \cos t \), and the general solution is \( \tilde{C} e^{-A(t)} = \tilde{C} e^{-\cos t} \).)

Substitute in the initial condition \( y(0) = \tilde{C} e^{-\cos 0} = 1 \), therefore \( \tilde{C} = e \). So the solution to the initial value problem is
\[
y(t) = e \times e^{-\cos t} = e^{1-\cos t}.
\]

5. (The logistic equation)

(a) Find the general solution to the equation \( y' = y(1-y) \).
We solve the ODE by separation of variables. Write the equation as \( \frac{dy}{y(1-y)} = dt \), integrating by partial fractions using
\[
\int \frac{dy}{y(1-y)} = \int \frac{dy}{y} + \int \frac{dy}{1-y} = \ln |y| - \ln |1-y| + C
\]
we get \( \ln \left| \frac{y}{1-y} \right| = t + C \), then exponentiate both sides to get \( \frac{y}{1-y} = \tilde{C}e^t \), \( \tilde{C} \neq 0 \). Then solve for \( y \) to get
\[
y = \frac{\tilde{C}e^t}{1 + \tilde{C}e^t}, \quad \tilde{C} \neq 0.
\]

Then in addition find constant solutions by setting right hand side \( y_0(1 - y_0) = 0 \) we have two more solutions: \( y = 0, y = 1 \). Combined with above we get the general solutions:
\[
y = \frac{\tilde{C}e^t}{1 + \tilde{C}e^t}, \quad \tilde{C} \text{ is any constant; or } y = 1.
\]

(b) Suppose \( y(0) = 1/2 \), give the solution to the initial value problem, and describe the behavior when \( t \) goes to \( \infty \)

Clearly \( y = 1 \) is not a solution. So substitute in \( y = \frac{\tilde{C}e^t}{1 + \tilde{C}e^t} \), we get \( y(0) = \frac{\tilde{C}}{1 + \tilde{C}} = 1/2 \) therefore \( \tilde{C} = 1 \). So the solution to the initial value problem is
\[
y = \frac{e^t}{1 + e^t}.
\]

As \( t \to \infty \), \( e^t \to \infty \), and \( y(t) \) is (increasingly) approaching 1.

(c) Suppose \( y(0) = 1 \), repeat (b)

\( y = 1 \) is the solution. (One should also check \( y = \frac{\tilde{C}e^t}{1 + \tilde{C}e^t} \) does not give a solution for any \( \tilde{C} \) by substitution.) As \( t \to 1 \), \( y(t) \) will still keep a constant 1.

(d) Suppose \( y(0) = 2 \), repeat (b)

Clearly \( y = 1 \) is not a solution. So substitute in \( y = \frac{\tilde{C}e^t}{1 + \tilde{C}e^t} \), we get \( y(0) = \frac{\tilde{C}}{1 + \tilde{C}} = 2 \) therefore \( \tilde{C} = -2 \). So the solution to the initial value problem is
\[
y = \frac{2e^t}{2e^t - 1}.
\]

As \( t \to \infty \), \( e^t \to \infty \), and \( y(t) \) is (decreasingly) approaching 1.

(e) In the population growth model, what do scenarios (b)–(d) mean?

Scenario (b): the starting population is smaller than the cap population, so the population will keep growing, but the growth rate is smaller and smaller, and the population will approach (but never reach) the limit 1.

Scenario (c): the starting population is already the maximum, so it will keep at the maximum.

Scenario (d): the starting population is already bigger than the maximum, therefore the population will decrease, with a slower and slower decreasing rate as it approaches (but never reaches) the limit 1.