18.03 RECITATION SHEET WEEK 12 – SOLUTIONS

1. The heat equation.

(a) Solve the following initial-boundary value problem for the heat equation:

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = 2\sin(3x) - 5\sin(4x).$$

Solution: In the lecture notes we saw the "theorem" that the general solution to the heat equation with Dirichlet boundary conditions,

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(L,t) = 0,$$

is given by the series

$$u(x,t) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{\pi k}{L}x\right) e^{-\nu\left(\frac{\pi k}{L}\right)^2 t}.$$

Here, $\nu = 2$ and $L = \pi$. We now need to match the coefficients b_k such that the initial condition is satisfied. At $t = 0$ the general formula reads

$$u(x,0) = \sum_{k=1}^{\infty} b_k \sin(kx).$$

This is a sine-Fourier series, and our initial condition is also given in the form of a sine-Fourier series. We just need to compare coefficients and find $b_3 = 2$, $b_4 = -5$, and all other $b_k = 0$. With this, the solution is

$$u(x,t) = 2e^{-18t}\sin(3x) - 5e^{-32t}\sin(4x).$$

(b) At what exponential rate does the solution to this equation go to 0 as $t \to \infty$?

Solution: The dominant term is $2e^{-18t}\sin(3x)$ so the rate of decay is 18.

2. The wave equation.

(a) Find the sine Fourier series on the interval $[0, 2\pi]$ for the function $f(x) = x$. (You will need to integrate by parts.)

Solution: The formula for the coefficients of the sine-Fourier series

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{\pi k}{L}x\right)$$

is

$$b_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{\pi k}{L}x\right)dx.$$

Here, $L = 2\pi$ and $f(x) = x$. Therefore,

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_0^{2\pi} x\sin\left(\frac{k}{2}x\right)dx \\ &= -\frac{2}{\pi k}\left(x\cos(kx/2)\right)_0^{2\pi} + \frac{2}{\pi k}\underbrace{\int_0^{2\pi}\cos(kx/2)dx}_{=0} \\ &= -\frac{2}{\pi k}\left(2\pi\cos(k\pi) - 0\cdot\cos(0)\right) \\ &= \frac{4}{k}(-1)^{k+1}. \end{aligned}$$

In the second line, we integrated by parts.

With this, the sine Fourier series is

$$f(x) = \sum_{k=1}^{\infty}\frac{4}{k}(-1)^{k+1}\sin\left(\frac{k}{L}x\right).$$

(b) Solve the following initial-boundary value problem for the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(2\pi,t) = 0, \quad u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = x.$$

Solution:

The general solution of the wave equation I-BVP with Dirichlet boundary conditions is

$$u(x,t) = \sum_{k=1}^{\infty}\left(b_k^{(1)}\cos\left(\frac{c\pi k}{L}t\right) + b_k^{(2)}\sin\left(\frac{c\pi k}{L}t\right)\right)\sin\left(\frac{\pi k}{L}x\right),$$

where $b_k^{(1)}$ are the sine-Fourier series coefficients of $u(x,0)$ and $(c\pi k/L)b_k^{(2)}$ are the sine-Fourier coefficients of $\partial u(x,0)/\partial t$. Here, the wave velocity is $c = 1$ and $L = 2\pi$.

Since $u(x,0) = 0$, all the $b_k^{(1)} = 0$. We know the sine-Fourier coefficients of $\partial u(x,0)/\partial t = x$ from part (a), so that

$$\frac{k}{2}b_k^{(2)} = \frac{4}{k}(-1)^{k+1} \Rightarrow b_k^{(2)} = \frac{8}{k^2}(-1)^{k+1}.$$

Plugging this back into the general solution we find

$$u(x,t) = \sum_{k=1}^{\infty}\frac{8}{k^2}(-1)^{k+1}\sin\left(\frac{k}{2}t\right)\sin\left(\frac{k}{2}x\right).$$

3. The heat equation redux.

(a) Find the cosine Fourier series on the interval $[0, \pi]$ for the function $f(x) = x$. (You will need to integrate by parts and consider the case $k = 0$ separately.)

Solution:
The cosine-Fourier series is defined as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi}{L}x\right),$$

and the coefficients are given by the formula

$$a_k = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{k\pi}{L}x\right)dx.$$

Here, $L = \pi$ and $f(x) = x$ (Remember that the function is periodically continued, so this is a "sawtooth" function).

We consider the case $k = 0$ first and calculate

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left(\frac{1}{2}x^2\right)_0^{\pi} = \pi.$$

For the other cases $k > 0$ we find

$$\begin{aligned}
a_k &= \frac{2}{\pi}\int_0^{\pi} x\cos(kx)dx \\
&= \frac{2}{\pi}\left(x\frac{1}{k}\underbrace{\sin(kx)}_{=0}\right)_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}\frac{1}{k}\sin(kx)\,dx \\
&= -\frac{2}{\pi k}\left(-\frac{1}{k}\cos(kx)\right)_0^{\pi} \\
&= \frac{2}{\pi k^2}((-1)^k - 1) \\
&= \begin{cases} -\frac{4}{\pi k^2}, & k \text{ odd} \\ 0, & k \text{ even,} \end{cases}
\end{aligned}$$

where we integrated by parts in the second line. Putting it all together, the cosine-Fourier series of $f(x) = x$ on $[0, \pi]$ is

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k \text{ odd}} \frac{1}{k^2}\cos(kx).$$

(b) (*) Using that the Fourier series in part (a) converges to the function $f(x) = x$ at $x = 0$, prove that the sum of the series $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is equal to $\frac{\pi^2}{6}$. (Hint: you will first compute the sum of the series over odd k only; then you need to express the sum of the even terms via the sum of the entire series. This is known as the Basel problem and was solved by Euler in 1734 by a different method.)

Solution:

Let $S = \sum_{k=1}^{\infty} \frac{1}{k^2}$. We first evaluate the sum over only the odd k. To do this, we consider the cosine-Fourier series for $f(x) = x$ from part (a) at $x = 0$ to find

$$\begin{aligned} 0 &= \frac{\pi}{2} - \frac{4}{\pi} \sum_{k \text{ odd}} \frac{1}{k^2} \cos(0) \\ \Rightarrow \sum_{k \text{ odd}} \frac{1}{k^2} &= \frac{\pi^2}{8}. \end{aligned}$$

Next, we calculate the sum over the even k,

$$\sum_{k \text{ even}} \frac{1}{k^2} = \sum_{j=1}^{\infty} \frac{1}{(2j)^2} = \frac{1}{4} \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{1}{4} S.$$

So, the sum over the even k is proportional to the whole sum! We now put it all together:

$$\begin{aligned} S &= \sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k \text{ odd}} \frac{1}{k^2} + \sum_{k \text{ even}} \frac{1}{k^2} \\ \Leftrightarrow S &= \frac{\pi^2}{8} + \frac{1}{4} S \\ \Leftrightarrow \left(1 - \frac{1}{4}\right) S &= \frac{\pi^2}{8} \\ \Leftrightarrow S &= \frac{\pi^2}{6}, \end{aligned}$$

which is what we wanted to show.

(c) Solve the following initial-boundary value problem for the heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi, t) = 0, \quad u(x,0) = x.$$

What is the behavior of the solution as $t \to \infty$?

Solution: This is the B-IVP for the Heat equation with Neumann boundary conditions. From the lecture notes we know the formula for the general solution,

$$u(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k e^{-\nu \left(\frac{\pi k}{L}\right)^2 t} \cos\left(\frac{\pi k}{L} x\right).$$

Here, $\nu = 1$ and $L = \pi$. The a_k are the cosine-Fourier series coefficients of the initial condition, which here is $u(x, 0) = x$. We have already calculated these coefficients in part (a), so we just need to plug them in now:

$$u(x,t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k \text{ odd}} \frac{1}{k^2} e^{-k^2 t} \cos(kx) .$$

For $t \to \infty$, all the exponentials in the series decay to 0 so that

$$u(x,t) \to \frac{\pi}{2} \qquad (t \to \infty).$$

The solution tends to a constant.

(d) A heated rod of length π initially has temperature x at each point x. We insulate the rod and its temperature $u(x, t)$ is governed by the heat equation from part (c) of this problem. Show that the temperature in the middle of the rod stays the same with time.

Solution:
We evaluate the temperature we calculated in part (c) at the midpoint $x = \pi/2$ of the rod,

$$\begin{aligned} u(\pi/2, t) &= \frac{\pi}{2} - \frac{4}{\pi} \sum_{k \text{ odd}} \frac{1}{k^2} e^{-k^2 t} \underbrace{\cos\left(k \frac{\pi}{2}\right)}_{=0} \\ &= \frac{\pi}{2}, \end{aligned}$$

where we used that $\cos(\pi k/2) = 0$ for all odd k. Thus, the temperature at the midpoint remains the same at all times.

4. Consider the system of nonlinear equations

$$y_1' = y_2, \quad y_2' = y_1^2 - 1.$$

(a) Find all the critical points of the system.

Solution:
The critical points are defined by $y_1' = 0$ and $y_2' = 0$. This immediately implies $y_2 = 0$ and $y_1 = \pm 1$. So the two critical points are $(y_1, y_2) = (1, 0)$ and $(y_1, y_2) = (-1, 0)$.

(b) Write down the linearized system at each critical point. Is it stable/semistable/unstable?

To find the linearized system we calculate the Jacobian in general for $F_1(y_1, y_2) = y_2$ and $F_2(y_1, y_2) = y_1^2 - 1$. It is

$$A(y_1, y_2) = \begin{pmatrix} \partial F_1/\partial y_1 & \partial F_1/\partial y_2 \\ \partial F_2/\partial y_1 & \partial F_2/\partial y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2y_1 & 0 \end{pmatrix}.$$

Next, we evaluate the Jacobian at the two critical points. First we consider $(y_1, y_2) = (1, 0)$. We find

$$A(1, 0) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

Near the critical point, the nonlinear system behaves like the linear system $\vec{y}' = A\vec{y}$, so stability depends on the eigenvalues of A. They are

$$\begin{aligned} \lambda_{1,2} &= \frac{\mathrm{tr}(A)}{2} \pm \sqrt{\frac{\mathrm{tr}(A)^2}{4} - \det(A)} \\ &= \pm\sqrt{2}. \end{aligned}$$

Since one of the eigenvalues has a positive real part, this critical point is unstable (in fact, it is a saddle point).

We now do the same at the other critical point $(y_1, y_2) = (-1, 0)$. The Jacobian is

$$A(-1, 0) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}.$$

Using the same formula again we see that the eigenvalues are $\lambda_{1,2} = \pm i\sqrt{2}$. Both of them have a real part of 0, so this critical point is semistable (in fact, it is a center).