18.03 RECITATION SHEET WEEK 12

1. The heat equation.

(a) Solve the following initial-boundary value problem for the heat equation:

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = 2\sin(3x) - 5\sin(4x).$$

(b) At what exponential rate does the solution to this equation go to 0 as $t \to \infty$?

2. The wave equation.

(a) Find the sine Fourier series on the interval $[0, 2\pi]$ for the function $f(x) = x$. (You will need to integrate by parts.)

(b) Solve the following initial-boundary value problem for the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(2\pi,t) = 0, \quad u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = x.$$

3. The heat equation redux.

(a) Find the cosine Fourier series on the interval $[0, \pi]$ for the function $f(x) = x$. (You will need to integrate by parts and consider the case $k = 0$ separately.)

(b) (*) Using that the Fourier series in part (a) converges to the function $f(x) = x$ at $x = 0$, prove that the sum of the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is equal to $\frac{\pi^2}{6}$. (Hint: you will first compute the sum of the series over odd k only; then you need to express the sum of the even terms via the sum of the entire series. This is known as the Basel problem and was solved by Euler in 1734 by a different method.)

(c) Solve the following initial-boundary value problem for the heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0, \quad u(x,0) = x.$$

What is the behavior of the solution as $t \to \infty$?

(d) A heated rod of length π initially has temperature x at each point x. We insulate the rod and its temperature $u(x,t)$ is governed by the heat equation from part (c) of this problem. Show that the temperature in the middle of the rod stays the same with time.

4. Consider the system of nonlinear equations

$$y_1' = y_2, \quad y_2' = y_1^2 - 1.$$

(a) Find all the critical points of the system.

(b) Write down the linearized system at each critical point. Is it stable/semistable/unstable?