18.03 RECITATION SHEET WEEK 9 SOLUTIONS

(Questions with * are optional)

1. Compute the exponential $\exp(tA)$, where $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. Verify by hand that $\vec{y}(t) = \exp(tA)\vec{w}$ solves the system of ODEs $\vec{y}' = A\vec{y}$ for any constant vector $\vec{w}$.

In order to calculate the matrix exponential $\exp(tA)$ *we diagonalize as* $A = SDS^{-1}$. *For that we need the eigenvalues and associated eigenvectors of* A. *We calculate the roots of the characteristic polynomial*

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & -1 & \lambda \end{vmatrix} = \lambda(\lambda^2 + 1) = 0.$$

We find the eigenvalues $\lambda_1 = 0$, $\lambda_2 = i$, $\lambda_3 = -i$. *The associated eigenvectors are given by the null space of* $\lambda I - A$. *We solve*

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & -1 & \lambda \end{pmatrix} \vec{v} = 0.$$

For $\lambda_1 = 0$, *the last two rows give* $v_2 = v_3 = 0$ *and the variable* v_1 *can be chosen freely. This gives* $\vec{v}_1 = (1, 0, 0)^T$. *For the rest of the eigenvalues the first row implies that* $v_1 = 0$ *and the second row gives*

$$\lambda v_2 + v_3 = 0 \Rightarrow v_2 = -v_3/\lambda.$$

Inserting eigenvalues λ_2 *and* λ_3 *gives* $\vec{v}_2 = (0, i, 1)^T$ *and* $\vec{v}_3 = (0, -i, 1)^T$. *Note that* $\vec{v}_3 = (\vec{v}_2)^*$. *Now,*

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \; S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & -i \\ 0 & 1 & 1 \end{pmatrix}.$$

Next we invert S. *Writing the augmented system gives*

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & i & -i & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right).$$

$R3 \to R3 + iR2$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & i & -i & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & i & 1 \end{array}\right).$$

$R1 \to -iR1$ and $R2 \to R2/2$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -i & 0 \\ 0 & 0 & 1 & 0 & i/2 & 1/2 \end{array}\right).$$

$R2 \to R2 + R3$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -i/2 & 1/2 \\ 0 & 0 & 1 & 0 & i/2 & 1/2 \end{array}\right).$$

Finally, we write the matrix exponential

$$e^{tA} = Se^{tD}S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & -i \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{it} & 0 \\ 0 & 0 & e^{-it} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -i/2 & 1/2 \\ 0 & i/2 & 1/2 \end{pmatrix}.$$

Calculating the first matrix product gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & ie^{it} & -ie^{-it} \\ 0 & e^{it} & e^{-it} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -i/2 & 1/2 \\ 0 & i/2 & 1/2 \end{pmatrix}$$

giving

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & (e^{it} + e^{-it})/2 & i(e^{it} - e^{-it})/2 \\ 0 & -i(e^{it} - e^{-it})/2 & (e^{it} + e^{-it})/2 \end{pmatrix} = \underline{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}}.$$

We check the solution. First we calculate

$$\vec{y}\,' = \left(\frac{\mathrm{d}}{\mathrm{d}t} \exp(tA)\right) \vec{w} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin t & -\cos t \\ 0 & \cos t & -\sin t \end{pmatrix} \vec{w}$$

We see if this works by calculating

$$A \exp(tA)\vec{w} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \vec{w} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin t & -\cos t \\ 0 & \cos t & -\sin t \end{pmatrix} \vec{w}.$$

It should be noted here that $\exp(tA)$ *gives a rotation by angle t around the x-axis. In fact, all the rotations in $\mathbb{R}^3$ can be expressed as* $\exp(tB)$ *with some matrix B.*

2. In this problem we solve the system of ODEs $y_1' = y_2 + 1$, $y_2' = y_1 + 2$ using variation of parameters.

(a) Write the system in the vector form $\vec{y}' = A\vec{y} + \vec{b}(t)$.

The first row of A is $(0,1)$ and the second row is $(1,0)$ while $\vec{b}$ is given by the constant coefficients 1 and 2. We get

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(b) Compute the matrix exponential e^{tA}.

As in the previous problem, we find the eigenvalues and eigenvectors. The roots of the characteristic polynomial are found from

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

giving $\lambda_1 = 1$ and $\lambda_2 = -1$. We find the eigenvectors from

$$\begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix} \vec{v} = 0.$$

The first row gives $v_2 = v_1 \lambda$. Plugging in the eigenvalues we find the eigenvectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We have

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We calculate the inverse of S. The augmented system is

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right).$$

$R2 \to R2 - R1:$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right).$$

$R2 \to -R2/2$ *followed by* $R1 \to R1 - R2:$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right).$$

Finally we can calculate the exponential

$$\exp(tA) = S \exp(tD) S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}.$$

Calculating this product we find

$$\exp(tA) = \underline{\frac{1}{2} \begin{pmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{pmatrix}}.$$

(c) We will look for the solution in the form $\vec{y}(t) = e^{tA}\vec{u}(t)$. Write the equation for $\vec{u}\,'(t)$.

Plugging $\vec{y}$ in the original equation gives

$$\vec{y}\,' = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}\vec{u}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t}e^{tA}\right)\vec{u} + e^{tA}\frac{\mathrm{d}}{\mathrm{d}t}\vec{u}.$$

This gives

$$\vec{y}\,' = Ae^{tA}\vec{u} + e^{tA}\vec{u}\,' = A\vec{y} + \vec{b}.$$

Plugging in $\vec{y}(t) = e^{tA}\vec{u}(t)$ gives

$$e^{tA}\vec{u}\,' = \vec{b}.$$

We can invert e^{tA} to give

$$\vec{u}\,' = e^{-tA}\vec{b}. = \frac{1}{2}\begin{pmatrix} e^t + e^{-t} & -e^t + e^{-t} \\ -e^t + e^{-t} & e^t + e^{-t} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \underline{\frac{1}{2}\begin{pmatrix} 3e^{-t} - e^t \\ 3e^{-t} + e^t \end{pmatrix}}$$

(d) Integrate to find $\vec{u}(t)$.

Integrating term by term gives

$$\vec{u} = \frac{1}{2}\begin{pmatrix} -3e^{-t} - e^t \\ -3e^{-t} + e^t \end{pmatrix} + \vec{c} = \frac{1}{2}\begin{pmatrix} -3e^{-t} - e^t + 2c_1 \\ -3e^{-t} + e^t + 2c_2 \end{pmatrix},$$

where $\vec{c} = (c_1, c_2)^T$ is an arbitrary constant.

(e) Find the general solution $\vec{y}(t)$ to the system.

We have

$$\vec{y} = e^{tA}\vec{u} = \frac{1}{4}\begin{pmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{pmatrix}\begin{pmatrix} -3e^{-t} - e^t + 2c_1 \\ -3e^{-t} + e^t + 2c_2 \end{pmatrix}.$$

Calculating the entries gives

$$\begin{cases} y_1 = -2 + \frac{c_1}{2}\left(e^t + e^{-t}\right) + \frac{c_2}{2}\left(e^t - e^{-t}\right), \\ y_2 = -1 + \frac{c_2}{2}\left(e^t + e^{-t}\right) + \frac{c_1}{2}\left(e^t - e^{-t}\right). \end{cases}$$

We can redefine $C_1 = c_1 + c_2$ and $C_2 = c_1 - c_2$ to find

$$\begin{cases} y_1 = -2 + C_1e^t + C_2e^{-t}, \\ y_2 = -1 + C_1e^t - C_2e^{-t}. \end{cases}$$

3. In this problem we follow the method of Section 9.2 in the lecture notes to find the Neumann eigenvalues and eigenfunctions for D^2 on the interval $[0, \pi]$. Namely we will find all complex numbers λ such that there exists a nonzero function $y(x)$ on the interval $[0, \pi]$ satisfying $y'' = \lambda y$, $y'(0) = y'(\pi) = 0$.

(a) Use integration by parts to show that each eigenvalue λ has to be a nonpositive real number.

We follow the proof from the lecture notes for Dirichlet boundary conditions. Multiplying $y'' = \lambda y$ by $\bar{y}$ (the complex conjugate of y) on both sides and integrating from 0 to π gives

$$\int_0^\pi \bar{y}y''\mathrm{d}x = \int_0^\pi \lambda|y|^2\mathrm{d}x = \lambda \underbrace{\int_0^\pi |y|^2\mathrm{d}x}_{>0}\,. \tag{1}$$

Here we use $\bar{y}y = |y|^2$ and $\int_0^\pi |y|^2\,\mathrm{d}x > 0$ since y is not identically 0. On the other hand we integrate the left hand side of Eq. (1) by parts giving

$$\int_0^\pi \bar{y}y''\mathrm{d}x = \bar{y}y'\Big|_0^\pi - \int_0^\pi |y'|^2\mathrm{d}x.$$

The first part gives 0 because $y'(0) = y'(\pi) = 0$ and the second part $-\int_0^\pi |y'|^2\mathrm{d}x$ is ≤ 0. Now combining with Eq. (1) it implies that $\lambda \leq 0$. This also shows that λ is real.

(b) Find all λ (eigenvalues) and for each of them find a corresponding function y (eigenfunctions).

We start by writing the general solution for y. First we consider the special case $\lambda = 0$. Now $y'' = 0$, which is solved by

$$y = c_1 + c_2x.$$

The derivative is given by

$$y' = c_2.$$

Plugging this into the boundary conditions implies that $c_2 = 0$ so the eigenfunction corresponding to the eigenvalue $\lambda = 0$ is $y_0 = 1$ (just as eigenvectors, eigenfunctions are unique up to a scalar multiplying the eigenfunction).

Next we assume $\lambda < 0$. The general solution is

$$y = c_1\cos(\omega x) + c_2\sin(\omega x),$$

where $\omega = \sqrt{-\lambda}$. The differential is given by

$$y' = -c_1\omega\sin(\omega x) + c_2\omega\cos(\omega x).$$

The first boundary condition yields

$$y'(0) = c_2\omega = 0$$

showing that $c_2 = 0$. The second boundary condition gives

$$y'(\pi) = -c_1\omega\sin(\omega\pi) = 0,$$

which is true if and only if ω is an integer k. The values k and $-k$ correspond to the same eigenfunction so we can choose $k \geq 1$ without a loss of generality. To summarize we have eigenfunctions

$$y_k = \cos(kx),$$

where $k = 0, 1, ...$ corresponding to eigenvalues

$$\lambda_k = -k^2.$$

Note that $k = 0$ corresponds to the special case $\lambda = 0$ we covered in the beginning.