18.03 RECITATION SHEET WEEK 11

(Questions with * are optional)

1. Compute the exponential $\exp(tA)$, where
   $$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$ Verify by hand that $\vec{y}(t) = \exp(tA)\vec{w}$ solves the system of ODEs $\vec{y}' = A\vec{y}$ for any constant vector $\vec{w}$.

2. In this problem we solve the system of ODEs $y_1' = y_2 + 1$, $y_2' = y_1 + 2$ using variation of parameters.
   (a) Write the system in the vector form $\vec{y}' = A\vec{y} + \vec{b}(t)$.
   (b) Compute the matrix exponential $e^{tA}$.
   (c) We will look for the solution in the form $\vec{y}(t) = e^{tA}\vec{u}(t)$. Write the equation for $\vec{u}''(t)$.
   (d) Integrate to find $\vec{u}(t)$.
   (e) Find the general solution $\vec{y}(t)$ to the system.

3. In this problem we follow the method of Section 9.2 in the lecture notes to find the Neumann eigenvalues and eigenfunctions for $D^2$ on the interval $[0, \pi]$. Namely we will find all complex numbers $\lambda$ such that there exists a nonzero function $y(x)$ on the interval $[0, \pi]$ satisfying
   $$y'' = \lambda y, \quad y'(0) = y'(\pi) = 0.$$
   (a) Use integration by parts to show that each eigenvalue $\lambda$ has to be a nonpositive real number.
   (b) Find all $\lambda$ (eigenvalues) and for each of them find a corresponding function $y$ (eigenfunctions).