18.03 RECITATION SHEET WEEK 10 SOLUTIONS

(Questions with * are optional)

(1) (a) Write a 1st order companion system for the system of ordinary differential equations $y_1'' = 3y_1 - y_2$, $y_2''' = y_1' + 2y_2'$. (Hint: the entries of the vector $\vec{y}$ will be $y_1, y_1', y_2, y_2', y_2''$.)

a. We write $\vec{y} = (y_1, y_1', y_2, y_2', y_2'')$. We will write $v_1 = y_1', v_2 = y_2'$, $v_3 = y_2''$. Let us find the matrix expressing the components of $\vec{y}'$ in terms of the y_1, v_1, y_2, v_2, v_3. By definition, $y_1' = v_1$, so the first row should be $\begin{pmatrix}0 & 1 & 0 & 0 & 0\end{pmatrix}$. The formula $y_1'' = 3y_1 - y_2$, gives the second row $\begin{pmatrix}3 & 0 & -1 & 0 & 0\end{pmatrix}$. By definition, $y_2' = v_2$, giving the third row $\begin{pmatrix}0 & 0 & 0 & 1 & 0\end{pmatrix}$. By definition $y_2'' = v_3$, so the fourth row is $\begin{pmatrix}0 & 0 & 0 & 0 & 1\end{pmatrix}$. Finally, the equation for y_2''' gives the last row, as follows, and so $\vec{y}' = A\vec{y}$, where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \end{pmatrix}$$

(2) Consider the linear SEIR system $E' = \beta I - \alpha E$, $I' = \alpha E - \gamma I$, $R' = \gamma I$. We fix $\alpha := 0.2$, $\beta = 0.2$, $\gamma = 0.5$. (This is supposed to represent COVID-19 with extreme social distancing: $\beta = 0.2$ means 1 contact per person every 5 days).

(a) Write the model in the form $\vec{y}' = A\vec{y}$.

a. If $\vec{y} = \begin{pmatrix} E \\ I \\ R \end{pmatrix}$, then $\vec{y}' = A\vec{y}$ where

$$A = \begin{pmatrix} -0.2 & 0.2 & 0 \\ 0.2 & -0.5 & 0 \\ 0 & 0.5 & 0 \end{pmatrix}.$$

Indeed, the equation for E' gives the first row, the equation for I' gives the second row, and the equation for R' gives the last row.

(b) Find the eigenvalues and eigenvectors of A.

b. We calculate the determinant of the matrix $\det(\lambda I - A)$, where

$$\lambda I - A = \begin{pmatrix} \lambda + 0.2 & -0.2 & 0 \\ -0.2 & \lambda + 0.5 & 0 \\ 0 & -0.5 & \lambda \end{pmatrix}.$$

Expanding the determinant along the top row of $\lambda I - A$ gives:

$$= (\lambda + 0.2) \begin{vmatrix} \lambda + 0.5 & 0 \\ -0.5 & \lambda \end{vmatrix} - (-0.2) \cdot \begin{vmatrix} -0.2 & 0 \\ -0.5 & \lambda \end{vmatrix} + 0.$$

That simplifies to $\lambda^3 + 0.7\lambda^2 + 0.1\lambda - 0.04\lambda$. We can factor out the λ, so

$$\det(\lambda I - A) = \lambda(\lambda^2 + 0.7\lambda + 0.06).$$

Applying the quadratic formula to the second factor in the previous line, we obtain that the eigenvalues are

$$\lambda_3 = 0, \qquad \lambda_2 = -0.1, \qquad \lambda_1 = -0.6$$

We find an eigenvector for λ_3 by putting A in row-echelon form, as:

$$\begin{pmatrix} 0.2 & -0.2 & 0 \\ 0 & -0.3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector.

Next, we plug in λ_2 to obtain the matrix

$$\lambda_2 I - A = \begin{pmatrix} 0.1 & -0.2 & 0 \\ -0.2 & 0.4 & 0 \\ 0 & -0.5 & -0.1 \end{pmatrix}.$$

The second row is a multiple of the first row, and then subtract 2/5 or the third row from the first row to obtain:

$$\begin{pmatrix} 0.1 & 0 & 0.04 \\ 0 & 0 & 0 \\ 0 & -0.5 & -0.1 \end{pmatrix}.$$

Then x_3 is a free variable, and we obtain $5x_2 = -x_3$, and $5x_1 = -2x_3$. Then $\vec{v}_2 = (2, 1, -5)$ is an eigenvector for λ_2

Finally, for λ_1, we have

$$\lambda_1 I - A = \begin{pmatrix} -0.4 & -0.2 & 0 \\ -0.2 & -0.1 & 0 \\ 0 & -0.5 & -0.6 \end{pmatrix}.$$

Subtracting $\frac{1}{2}R1$ from $R2$, we obtain a row of all zeros. Then we subtract $\frac{2}{5}$ of $R3$ from $R1$ to obtain:

$$\begin{pmatrix} -0.4 & 0 & \frac{2}{5}(0.6) \\ 0 & 0 & 0 \\ 0 & -0.5 & -0.6 \end{pmatrix}.$$

Now, x_3 is a free variable, and $-0.5x_2 = 0.6x_3$, while $-0.4x_1 = -\frac{2}{5}(0.6)x_3$. If we set $x_3 = 5$, we then obtain $x_2 = -6$ and $(-0.4)x_1 = (-1.2)$, so $x_1 = 3$. That is, $\vec{v}_1 = (3, -6, 5)$ is an eigenvector for λ_1.

(c) Find the general solution to the system

c. The general solution is then

$$\begin{pmatrix} E(t) \\ I(t) \\ R(t) \end{pmatrix} = e^{-(0.6)t}C_1\vec{v}_1 + e^{-(0.1)t}C_2\vec{v}_2 + C_3\vec{v}_3 \qquad (1)$$

(d) Is it always true that $E(t) \to 0$? What is the fastest rate of exponential decay for $E(t)$ you can guarantee for every solution?

d. To determine if $E(t)$ always goes to zero, we note that the E-coordinate of v_3 above is zero, so that $E(t)$ in the general solution (1) is $3C_1e^{-(0.6)t} + 2C_2e^{-(0.1)t}$. Then the solution decreases exponentially with time, so $E(t) \to 0$ for any initial conditions, depending on C_1, C_2, C_3.
The fastest rate of decay that we can *guarantee* for $E(t)$ is that it decays like $e^{-(0.1)t}$.

(3) Consider a system of 2 weights of mass m each lying on a line between two walls. Each of the weights is connected to a wall (the left weight to the left wall, the right weight to the right wall) through a spring with coefficient k. The weights are connected to each other through a damper with coefficient b.

(a) Explain why the behavior of the displacements from equilibrium of the two weights, $x_1(t)$ and $x_2(t)$, is modeled by the second order system of ODEs $mx_1'' = -kx_1 - b(x_1' - x_2')$, $mx_2'' = -kx_2 - b(x_2' - x_1')$.

a. We find the forces acting on the springs. By Hooke's law, the left spring experiences a restoring force of kx_1, where x_1 is its distance from an equilibrium position, similarly for x_2. The damper tends to resist relative motion between the two weights (i.e. if the distance between the weights is constant, the damper exerts no force), and acts by a force proportional to the difference in the velocities of the weights, so contributes a term $-b(x_1' - x_2')$ to mx_1'', and contributes $-b(x_2' - x_1')$ to mx_2''.

(b) Write the first order companion system $\vec{y}' = A\vec{y}$ where $\vec{y}$ has entries $y_1 = x_1, y_2 = {x'}_1, y_3 = x_2, y_4 = {x'}_2$.

b. As usual, the rules $x_1' = y_2$ and $x_2' = y_4$ determine the first and third row of the matrix as $\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$, respectively. The term $y_2' = -(k/m)y_1 - (b/m)y_2 + (b/m)y_4$ gives that the second row is $\begin{pmatrix} -\frac{k}{m} & -\frac{b}{m} & 0 & \frac{b}{m} \end{pmatrix}$, and similarly we determine the fourth row, so that

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m} & -\frac{b}{m} & 0 & \frac{b}{m} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{b}{m} & -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}.$$

(c) In the remaining parts of this problem we assume that $m = 1$, $k = 9$, $b = 5$. Verify that $\vec{v}_1 = \begin{pmatrix} 1 \\ 3i \\ 1 \\ 3i \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -3i \\ 1 \\ -3i \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$, $\vec{v}_4 = \begin{pmatrix} 1 \\ -9 \\ -1 \\ 9 \end{pmatrix}$ are eigenvectors of A and find the corresponding eigenvalues. The given 4 vectors form a basis but you do not need to verify that.

c. To find the eigenvalues, we will plug each of the vectors $\vec{v}_1, \ldots, \vec{v}_4$ into $Ax = \lambda x$ and see what value of λ satisfies this equation.
For $\vec{v}_1$, we obtain

$$A\vec{v}_1 = \begin{pmatrix} (1)3i \\ -9(1) - 5(3i) + 5(3i) \\ 3i \\ 5(3i) - 9(1) - 5(3i) \end{pmatrix} = \begin{pmatrix} 3i \\ -9 \\ 3i \\ -9 \end{pmatrix} = 3i \begin{pmatrix} 1 \\ 3i \\ 1 \\ 3i \end{pmatrix}.$$

Next, for $\vec{v}_2$:

$$A\vec{v}_2 = \begin{pmatrix} (1)(-3i) \\ -9(1) - 5(-3i) + 5(-3i) \\ -3i \\ 5(-3i) - 9(1) - 5(-3i) \end{pmatrix} = -3i \begin{pmatrix} 1 \\ -3i \\ 1 \\ -3i \end{pmatrix}.$$

For $\vec{v}_3$:

$$A\vec{v}_3 = \begin{pmatrix} (1)(-1) \\ -9(1) - 5(-1) + 5(1) \\ 1 \\ 5(-1) - 9(-1) - 5(1) \end{pmatrix} = -1 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

For $\vec{v}_4$:

$$A\vec{v}_4 = \begin{pmatrix} (1)(-9) \\ -9(1) - 5(-9) + 5(9) \\ 9 \\ 5(-9) - 9(-1) - 5(9) \end{pmatrix} = -9 \begin{pmatrix} 1 \\ -9 \\ -1 \\ 9 \end{pmatrix}.$$

So, $\lambda_1 = 3i, \lambda_2 = -3i, \lambda_3 = -1, \lambda_4 = -9$, and we have verified that each of $\vec{v}_1, \ldots, \vec{v}_4$ are eigenvectors.

(d) Find the general solution to the system of ODEs. It is enough to write down the formulas for x_1 and x_2, you do not need to write the formulas for x'_1 and x'_2.

d. Recall that to get the general real solution, we take only one of the conjugate pair $(3i, -3i)$ of eigenvalues, but take the real and imaginary parts of the associated solution. That is, the general solution is

$$C_1\mathrm{Re}(e^{3it}\vec{v}_1) + C_2\mathrm{Im}(e^{3it}\vec{v}_1) + e^{-t}C_3\vec{v}_3 + e^{-9t}C_4\vec{v}_4.$$

The x_1 component is $C_1\cos(3t) + C_2\sin(3t) + C_3e^{-t} + C_4e^{-9t}$, while the x_2 component is $C_1\cos(3t) + C_2\sin(3t) - C_3e^{-t} - C_4e^{-9t}$. Then the general solution is:

$$x_1(t) = C_1\cos(3t) + C_2\sin(3t) + C_3e^{-t} + C_4e^{-9t},$$
$$x_2(t) = C_1\cos(3t) + C_2\sin(3t) - C_3e^{-t} - C_4e^{-9t}.$$

(e) Describe the behavior of the solutions corresponding to individual terms in the general solution (i.e. when 3 out of the 4 constants are taken to be 0)

e. For each of the first two terms: the weights are moving together, oscillating with angular frequency 3. For the last two terms: the weights are converging exponentially fast to the equilibrium with rates 1 and 9.

(4) Consider the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(a) Find the eigenvalues of A. Is A diagonalizable?

a. We calculate directly that $\det(\lambda I - A) = \lambda^3$. Thus, the only eigenvalue of A is zero.
The eigenvectors v of A are then exactly the vectors in the nullspace of A. If A were diagonalizable, that would mean that there is a set of three linearly independent vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ so that $A\vec{v}_1 = A\vec{v}_2 = A\vec{v}_3 = 0$. However, A is already in RREF, and we can see that a vector $\vec{v} = (v_1, v_2, v_3)$ is in the nullspace if and only if $v_2 = v_3 = 0$. This means there is only one dimension of eigenvectors for the eigenvalue 0. Thus, the matrix A is not diagonalizable, since it does not have three linearly independent eigenvectors.

(b) Compute the exponential e^{tA} using the definition of the exponential as a series. Note: only the first few terms of the series will be nonzero.

b. We need to calculate $A, A^2, \ldots$. Let us first calculate A^2. It is a quick check to see that

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, let us calculate A^3. It is a quick check to see that:

$$A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $A^n = 0$ for all $n \geq 3$. Then $e^{tA} = I + At + t^2A^2/2$. Using the above calculation of A^2, we have:

$$e^{tA} = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$