18.03 RECITATION SHEET WEEK 10

(Questions with * are optional)

1) Write a 1st order companion system for the system of ordinary differential equations $y_1'' = 3y_1 - y_2$, $y_2''' = y_1' + 2y_2'$. (Hint: the entries of the vector $\vec{y}$ will be $y_1, y_1', y_2, y_2', y_2''$.)

2) Consider the linear SEIR system $E' = \beta I - \alpha E$, $I' = \alpha E - \gamma I$, $R' = \gamma I$. We fix $\alpha := 0.2$, $\beta = 0.2$, $\gamma = 0.5$. (This is supposed to represent COVID-19 with extreme social distancing: $\beta = 0.2$ means 1 contact per person every 5 days).

(a) Write the model in the form $\vec{y}\,' = A\vec{y}$.

(b) Find the eigenvalues and eigenvectors of A

(c) Find the general solution to the system.

(d) Is it always true that $E(t) \to 0$? What is the fastest rate of exponential decay for $E(t)$ you can guarantee for every solution?

3) Consider a system of 2 weights of mass m each lying on a line between two walls. Each of the weights is connected to a wall (the left weight to the left wall, the right weight to the right wall) through a spring with coefficient k. The weights are connected to each other through a damper with coefficient b.

(a) Explain why the behavior of the displacements from equilibrium of the two weights, $x_1(t)$ and $x_2(t)$, is modeled by the second order system of ODEs $mx_1'' = -kx_1 - b(x_1' - x_2')$, $mx_2'' = -kx_2 - b(x_2' - x_1')$.

(b) Write the first order companion system $\vec{y}\,' = A\vec{y}$ where $\vec{y}$ has entries $y_1 = x_1, y_2 = x_1', y_3 = x_2, y_4 = x_2'$

In the remaining parts of this problem we assume that $m = 1$, $k = 9$, $b = 5$.

(c) Verify that $\vec{v}_1 = \begin{pmatrix} 1 \\ 3i \\ 1 \\ 3i \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -3i \\ 1 \\ -3i \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$, $\vec{v}_4 = \begin{pmatrix} 1 \\ -9 \\ -1 \\ 9 \end{pmatrix}$ are eigenvectors of A and find the corresponding eigenvalues. The given 4 vectors form a basis but you do not need to verify that.

(d) Find the general solution to the system of ODEs. It is enough to write down the formulas for x_1 and x_2, you do not need to write the formulas for x_1' and x_2'.

(e) Describe the behavior of the solutions corresponding to individual terms in the general solution (i.e. when 3 out of the 4 constants are taken to be 0).

4) Consider the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(a) Find the eigenvalues of A. Is A diagonalizable?

(b) Compute the exponential e^{tA} using the definition of the exponential as a series. Note: only the first few terms of the series will be nonzero.