18.03 RECITATION SHEET WEEK 10

(Questions with * are optional)

1) Write a 1st order companion system for the system of ordinary differential equations $y''_1 = 3y_1 - y_2$, $y''_2 = y'_1 + 2y'_2$. (Hint: the entries of the vector $\vec{y}$ will be $y_1, y'_1, y_2, y'_2$.)

2) Consider the linear SEIR system $E' = \beta I - \alpha E$, $I' = \alpha E - \gamma I$, $R' = \gamma I$. We fix $\alpha := 0.2$, $\beta = 0.2$, $\gamma = 0.5$. (This is supposed to represent COVID-19 with extreme social distancing: $\beta = 0.2$ means 1 contact per person every 5 days).
   (a) Write the model in the form $\vec{y}' = A\vec{y}$.
   (b) Find the eigenvalues and eigenvectors of $A$
   (c) Find the general solution to the system.
   (d) Is it always true that $E(t) \to 0$? What is the fastest rate of exponential decay for $E(t)$ you can guarantee for every solution?

3) Consider a system of 2 weights of mass $m$ each lying on a line between two walls. Each of the weights is connected to a wall (the left weight to the left wall, the right weight to the right wall) through a spring with coefficient $k$. The weights are connected to each other through a damper with coefficient $b$.
   (a) Explain why the behavior of the displacements from equilibrium of the two weights, $x_1(t)$ and $x_2(t)$, is modeled by the second order system of ODEs $mx''_1 = -kx_1 - b(x'_1 - x'_2)$, $mx''_2 = -kx_2 - b(x'_2 - x'_1)$.
   (b) Write the first order companion system $\vec{y}' = A\vec{y}$ where $\vec{y}$ has entries $y_1 = x_1, y_2 = x'_1, y_3 = x_2, y_4 = x'_2$.
   In the remaining parts of this problem we assume that $m = 1$, $k = 9$, $b = 5$.
   (c) Verify that $\vec{v}_1 = \begin{pmatrix} 1 \\ 3i \\ 1 \\ 3i \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -3i \\ 1 \\ -3i \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$, $\vec{v}_4 = \begin{pmatrix} 1 \\ -9 \\ -1 \\ 9 \end{pmatrix}$ are eigenvectors of $A$ and find the corresponding eigenvalues. The given 4 vectors form a basis but you do not need to verify that.
   (d) Find the general solution to the system of ODEs. It is enough to write down the formulas for $x_1$ and $x_2$, you do not need to write the formulas for $x'_1$ and $x'_2$. 

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(e) Describe the behavior of the solutions corresponding to individual terms in the general solution (i.e. when 3 out of the 4 constants are taken to be 0).

4) Consider the matrix 
\[
A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\]

(a) Find the eigenvalues of \(A\). Is \(A\) diagonalizable?
(b) Compute the exponential \(e^{tA}\) using the definition of the exponential as a series. Note: only the first few terms of the series will be nonzero.