SUPPLEMENTARY GRAPHS FOR §10.3

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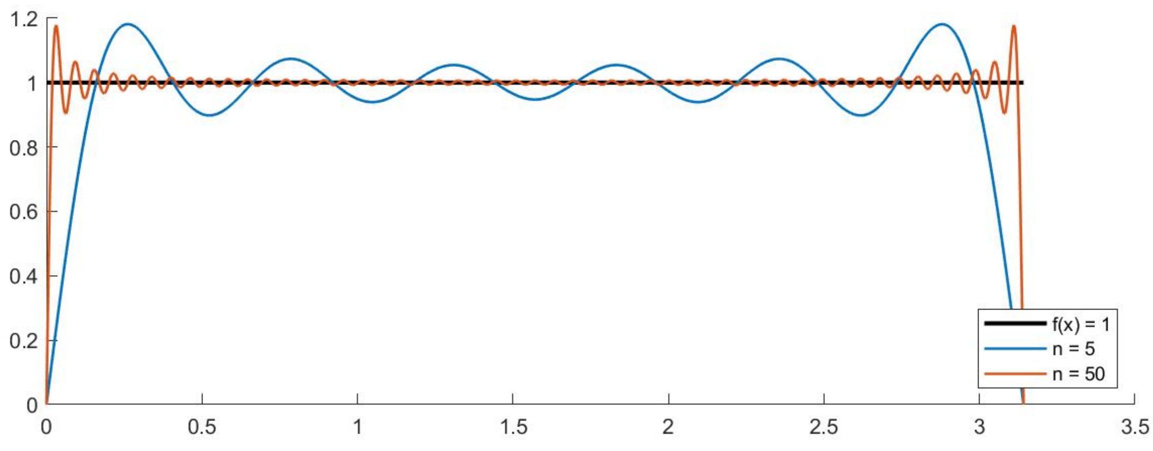
Consider the function

$$f(x) = 1, \quad 0 \le x \le \pi.$$

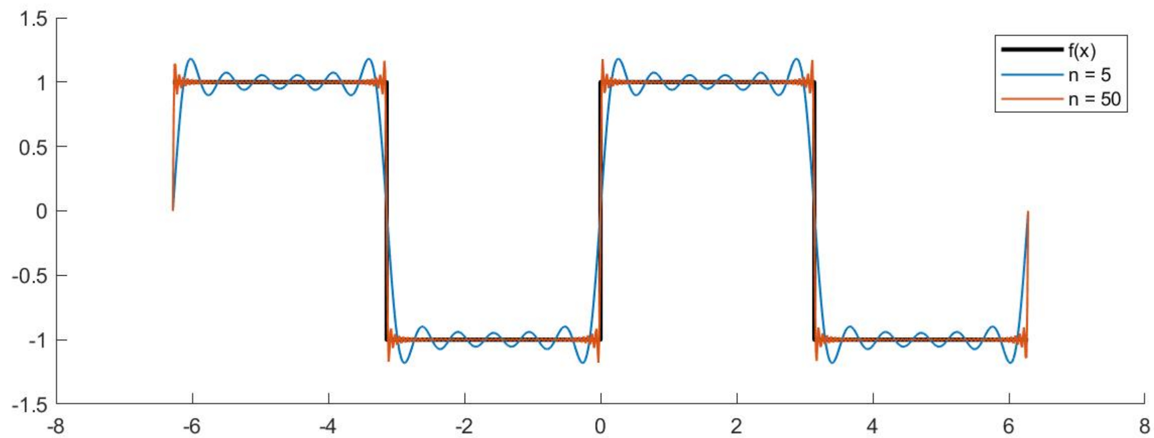
The corresponding sine Fourier series is

$$\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\sin((2j+1)x)}{2j+1} = \frac{4}{\pi}\sum_{k\ge 1 \text{ odd}}\frac{\sin(kx)}{k} = \frac{4}{\pi}\left(\frac{\sin x}{x} + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \cdots\right).$$

The figure below shows the partial sum of the above series over $j = 0, \dots, n$ for $n = 5$ and $n = 50$:



On the top next page we plot the partial sum of the same series for a larger range of values of x. Note that each term in the series (and thus the partial sum) is an odd 2π-periodic function. The sum of the series is the 2π-periodic odd extension of the function $f(x)$ except at the points $x = \pi n$ where it is equal to 0.



Next we look at the solution of the initial-boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t \geq 0,\ 0 \leq x \leq \pi;$$
$$u(0,t) = u(\pi,t) = 0, \quad t \geq 0;$$
$$u(x,0) = 1, \quad 0 < x < \pi$$

which is given by

$$u(x,t) = \frac{4}{\pi}\sum_{j=0}^{\infty} \frac{e^{-(2j+1)^2 t}\sin((2j+1)x)}{2j+1}.$$

Note that the profile of the solution approaches an (exponentially decaying) sine wave because the term with $j = 0$ dominates the series when $t \to \infty$.