

§9.2. Eigenvalues & eigenfunctions of differential operators

Previously we studied the eigen-equation for a matrix A : $A\vec{v} = \lambda\vec{v}$

Now we will study the eigen-equation for a 2nd order differential operator $P(D)$:

(*) $P(D)y = \lambda y$ THEORY

where y is a function and λ is a (complex) number
 If we just used the equation (*) then any number λ would be an eigenvalue and the corresponding eigenspace will always be 2-dimensional (since (*) always has a 2-dimensional space of solutions).

However we also impose homogeneous boundary conditions:

(**) $\begin{cases} P(D)y = \lambda y, & y = y(x), & x_1 \leq x \leq x_2 \\ y(x_1) = 0 \\ y(x_2) = 0 \end{cases} \rightarrow$ these are called Dirichlet boundary conditions

Definition: if λ, y solve (**)

and $y \neq 0$ (i.e. y is not equal to 0 everywhere)
then we say that

- λ is an eigenvalue of $P(D)$ on the interval $[x_1, x_2]$ with Dirichlet boundary conditions
 - y is an eigenfunction of $P(D)$ corresponding to λ
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In this course we only study a few eigenvalue/eigenfunction problems (we use them in the next chapter)

We will focus on the case $P(z) = z^2$, i.e.

$$P(D) = D^2$$

Goal: find the eigenvalues & eigenfunctions of the operator D^2 on the interval $[0, L]$ (for given $L > 0$) with Dirichlet boundary conditions

Step 1: Write the corresponding boundary value problem

$$\begin{cases} y'' = \lambda y \\ y(0) = 0 \\ y(L) = 0 \end{cases}$$

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Step 2: Show that for λ to be an eigenvalue we need $\lambda \leq 0$

A classical proof of this is by integration by parts (IBP):

multiply the equation $y'' = \lambda y$ by \bar{y} (the complex conjugate of y) to get

$$y''(x) \bar{y}(x) = \lambda |y(x)|^2$$

Now integrate on the interval $[0, L]$:

$$\begin{aligned} \lambda \int_0^L |y(x)|^2 dx &= \int_0^L \lambda |y(x)|^2 dx = \int_0^L y''(x) \bar{y}(x) dx \\ &= \int_0^L (y'(x))' \bar{y}(x) dx \stackrel{\text{IBP}}{=} \underbrace{y'(x) \bar{y}(x)}_{\substack{0 \text{ since} \\ y(0) = y(L) = 0}} \Big|_{x=0}^L - \int_0^L y'(x) \bar{y}'(x) dx \\ &= - \int_0^L |y'(x)|^2 dx. \end{aligned}$$

We ended up with

$$\lambda \int_0^L |y(x)|^2 dx = - \int_0^L |y'(x)|^2 dx.$$

Since $y \neq 0$ we have $\int_0^L |y(x)|^2 dx > 0$.

Also, $\int_0^L |y'(x)|^2 dx \geq 0$.

Therefore $\lambda \leq 0$

Step 3: find the eigenvalues. Assume $\lambda \leq 0$

General solution to $y'' = \lambda y$ is

$$y(x) = C_1 y_1(x) + C_2 y_2(x) \text{ where}$$

$$\lambda = 0 \Rightarrow \begin{cases} y_1(x) = 1 \\ y_2(x) = x \end{cases}; \quad \lambda < 0 \Rightarrow \begin{cases} y_1(x) = \cos(\omega x) \\ y_2(x) = \sin(\omega x) \end{cases}$$

$\omega = \sqrt{-\lambda}$

Recall from §9.1 that the homogeneous boundary value problem has a $\neq 0$ solution \Leftrightarrow

$\Leftrightarrow \det W = 0$ where

$$W = \begin{pmatrix} y_1(0) & y_2(0) \\ y_1(L) & y_2(L) \end{pmatrix}$$

Now, if $\lambda = 0$ then $W = \begin{pmatrix} 1 & 0 \\ 1 & L \end{pmatrix}$

$\det W = L \neq 0 \Rightarrow 0$ is not an eigenvalue

Now assume that $\lambda < 0$, writing $\lambda = -\omega^2$ where $\omega > 0$

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Then $y_1(x) = \cos(\omega x)$, $y_2(x) = \sin(\omega x)$, so

$$W = \begin{pmatrix} 1 & 0 \\ \cos(\omega L) & \sin(\omega L) \end{pmatrix}, \det W = \sin(\omega L).$$

Thus λ is an eigenvalue $\Leftrightarrow \det W = 0$

$$\Leftrightarrow \sin(\omega L) = 0 \Leftrightarrow \omega = \frac{\pi k}{L}, \quad k \geq 1 \text{ integer.}$$

So the eigenvalues are

$$\lambda_k = -\left(\frac{\pi k}{L}\right)^2 \text{ where } k \geq 1 \text{ integer}$$

$$\text{i.e. } \lambda = -\left(\frac{\pi}{L}\right)^2, -4\left(\frac{\pi}{L}\right)^2, -9\left(\frac{\pi}{L}\right)^2, \dots$$

Note: there are infinitely many eigenvalues

Step 4: find the eigenfunctions

Put $\lambda = -\left(\frac{\pi k}{L}\right)^2$, then the general

solution to $y'' = \lambda y$ is

$$y = C_1 \cos\left(\frac{\pi k}{L} x\right) + C_2 \sin\left(\frac{\pi k}{L} x\right)$$

Plug in the boundary conditions:

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$$\begin{cases} 0 = y(0) = C_1 \\ 0 = y(L) = C_1 \cos(\pi k) + C_2 \sin(\pi k) = (-1)^k C_1. \end{cases}$$

So $C_1 = 0 \Rightarrow$ a basis of eigenfunctions with eigenvalue $\lambda_k = -\left(\frac{\pi k}{L}\right)^2$ is

$$y_k(x) = \sin\left(\frac{\pi k}{L}x\right)$$

Illustration for $L = \pi$:



