§9.2. Eigenvalues & eigenfunctions of differential operators

Previously we studied the eigen-equation for a matrix A: $\boxed{A\vec{v} = \lambda \vec{v}}$

Now we will study the eigen-equation for a 2^{nd} order differential operator $P(D)$:

(*) $\boxed{P(D)y = \lambda y}$ $\boxed{\text{THEORY}}$

where y is a function and λ is a (complex) number

If we just used the equation (*) then any number λ would be an eigenvalue and the corresponding eigenspace will always be 2-dimensional (since (*) always has a 2-dimensional space of solutions).

However we also impose homogeneous boundary conditions:

$$(**)\quad \begin{cases} P(D)y = \lambda y, \quad y = y(x), \quad x_1 \le x \le x_2 \\ y(x_1) = 0 \\ y(x_2) = 0 \end{cases}$$

these are called Dirichlet boundary conditions

Definition: if λ, y solve $(**)$ and $y \neq 0$ (i.e. y is not equal to 0 everywhere) then we say that

- λ is an eigenvalue of $P(D)$ on the interval $[x_1, x_2]$ with Dirichlet boundary conditions
- y is an eigenfunction of $P(D)$ corresponding to λ

In this course we only study a few eigenvalue/eigenfunction problems (we use them in the next chapter)

We will focus on the case $P(z) = z^2$, i.e.

$$P(D) = D^2$$

Goal: find the eigenvalues & eigenfunctions of the operator D^2 on the interval $[0, L]$ (for given $L > 0$) with Dirichlet boundary conditions

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Step 1: write the corresponding boundary value problem

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$$\begin{cases} y'' = \lambda y \\ y(0) = 0 \\ y(L) = 0 \end{cases}$$

Step 2: Show that ~~for~~ λ to be an eigenvalue we need $\boxed{\lambda \leq 0}$

A classical proof of this is by integration by parts (IBP):

multiply the equation $y'' = \lambda y$ by $\bar{y}$ (the complex conjugate of y) to get

$$y''(x)\overline{y(x)} = \lambda |y(x)|^2.$$

Now integrate on the interval $[0, L]$:

$$\lambda \int_0^L |y(x)|^2 dx = \int_0^L \lambda |y(x)|^2 dx = \int_0^L y''(x)\overline{y(x)}\, dx$$

$$= \int_0^L (y'(x))' \overline{y(x)}\, dx \overset{IBP}{=} \underbrace{y'(x)\overline{y(x)}\Big|_{x=0}^{L}}_{\substack{= \\ 0 \text{ since} \\ y(0)=y(L)=0}} - \int_0^L y'(x)\overline{y'(x)}\, dx$$

$$= -\int_0^L |y'(x)|^2 dx.$$

We ended up with

$$\lambda \cdot \int_0^L |y(x)|^2 dx = -\int_0^L |y'(x)|^2 dx.$$

Since $y \neq 0$ we have $\int_0^L |y(x)|^2 dx > 0$.

Also, $\int_0^L |y'(x)|^2 dx \geq 0$.

Therefore $\boxed{\lambda \leq 0}$

<u>Step 3</u>: find the eigenvalues. Assume $\lambda \leq 0$

General solution to $y'' = \lambda y$ is

$y(x) = C_1 y_1(x) + C_2 y_2(x)$ where

$\lambda = 0 \Rightarrow \begin{cases} y_1(x) = 1 \\ y_2(x) = x \end{cases}$; $\lambda < 0$ $(\omega = \sqrt{-\lambda})$ $\Rightarrow \begin{cases} y_1(x) = \cos(\omega x) \\ y_2(x) = \sin(\omega x) \end{cases}$

Recall from §9.1 that the homogeneous boundary value problem has a $\neq 0$ solution $\Leftrightarrow$

$\Leftrightarrow \det W = 0$ where

$$W = \begin{pmatrix} y_1(0) & y_2(0) \\ y_1(L) & y_2(L) \end{pmatrix}$$

Now, if $\lambda = 0$ then $W = \begin{pmatrix} 1 & 0 \\ 1 & L \end{pmatrix}$

$\det W = L \neq 0 \Rightarrow 0$ is <u>not</u> an eigenvalue

Now assume that $\lambda < 0$, writing
$\lambda = -\omega^2$ where $\omega > 0$

Then $y_1(x) = \cos(\omega x)$, $y_2(x) = \sin(\omega x)$, so

$$W = \begin{pmatrix} 1 & 0 \\ \cos(\omega L) & \sin(\omega L) \end{pmatrix}, \quad \det W = \sin(\omega L).$$

Thus λ is an eigenvalue $\Leftrightarrow \det W = 0 \Leftrightarrow$
$\Leftrightarrow \sin(\omega L) = 0 \Leftrightarrow \omega = \frac{\pi k}{L}$, $k \geq 1$ integer.

So the <u>eigenvalues</u> are

$$\boxed{\lambda_k = -\left(\frac{\pi k}{L}\right)^2 \text{ where } k \geq 1 \text{ integer}}$$

i.e. $\lambda = -\left(\frac{\pi}{L}\right)^2, -4\left(\frac{\pi}{L}\right)^2, -9\left(\frac{\pi}{L}\right)^2, \ldots$

Note: there are <u>infinitely many eigenvalues</u>

<u>Step 4</u>: find the eigenfunctions

Put $\lambda = -\left(\frac{\pi k}{L}\right)^2$, then the general solution to $y'' = \lambda y$ is

$$y = C_1 \cos\left(\frac{\pi k}{L} x\right) + C_2 \sin\left(\frac{\pi k}{L} x\right)$$

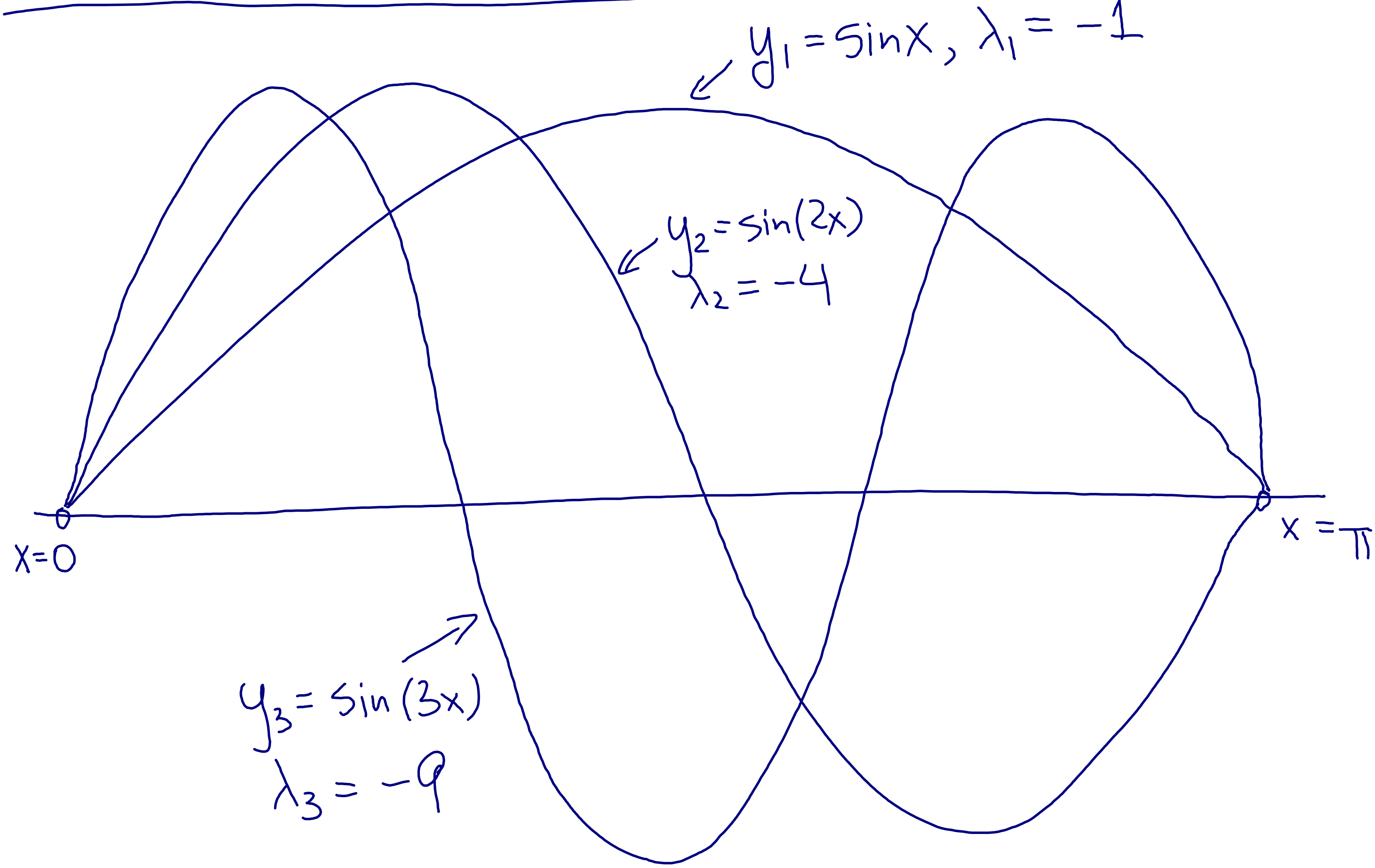
Plug in the boundary conditions:

$$\begin{cases} 0 = y(0) = C_1 \\ 0 = y(L) = C_1\cos(\pi k) + C_2\sin(\pi k) = (-1)^k C_1. \end{cases}$$

So $C_1 = 0$ $\Rightarrow$ a basis of eigenfunctions with eigenvalue $\lambda_k = -\left(\frac{\pi k}{L}\right)^2$ is

$$\boxed{y_k(x) = \sin\left(\frac{\pi k}{L}x\right)}$$

Illustration for $L = \pi$:



One can study an eigenvalue problem for the same operator with <u>different boundary conditions</u> and obtain <u>different eigenvalues and eigenfunctions</u>.

<u>Example:</u>

$$\begin{cases} y'' = \lambda y \\ y'(0) = 0 \\ y'(L) = 0 \end{cases}$$

these are called <u>Neumann</u> <u>boundary conditions</u>

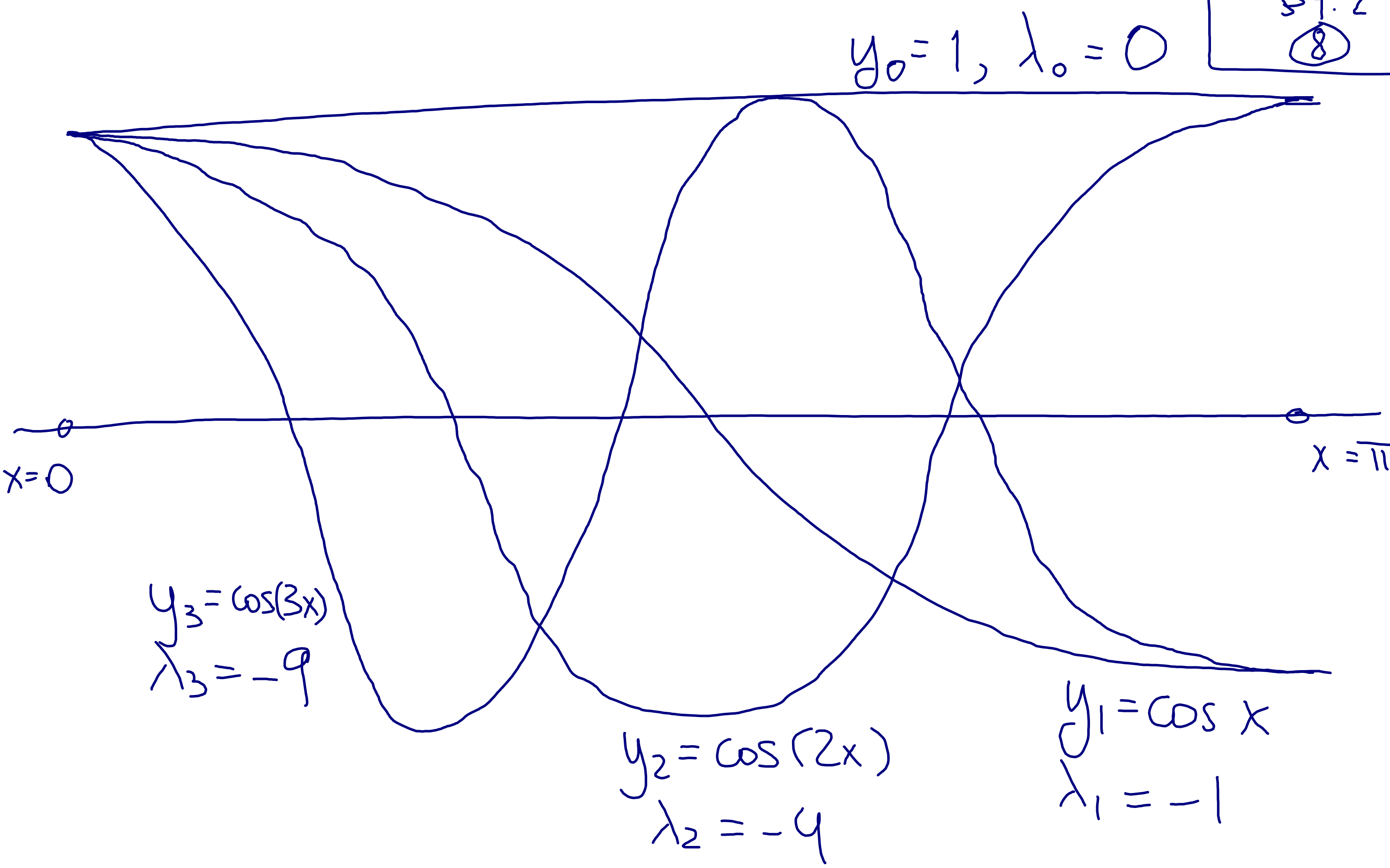
The eigenvalues will be $\lambda_k = -\left(\frac{\pi k}{L}\right)^2$ where $\boxed{k \geq 0}$ an integer

(so now 0 <u>is</u> an eigenvalue)

and the eigenfunctions are

$$y_1 = 1; \quad y_k(x) = \cos\left(\frac{\pi k}{L} x\right), \quad k \geq 1$$

Illustration on the next page (still for $L = \pi$):


$y_0 = 1$, $\lambda_0 = 0$
$x=0$
$x=\pi$
$y_3 = \cos(3x)$
$\lambda_3 = -9$
$y_2 = \cos(2x)$
$\lambda_2 = -4$
$y_1 = \cos x$
$\lambda_1 = -1$