§9. Boundary value problems

In this chapter we go back to studying a single ODE but with boundary conditions instead of initial conditions.

These will be useful in the study of partial differential equations (PDE) in the next chapter.

§9.1. Basic properties

We study a linear homogeneous 2nd order constant coefficient ODE

THEORY

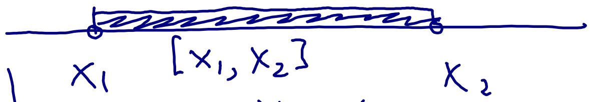
$$\boxed{P(D)y = 0} \quad \text{where } P(z) = a_2 z^2 + a_1 z + a_0, \quad a_2 \neq 0$$

(boundary conditions can be imposed for much more general ODEs but we restrict ourselves to this special class)

! CHANGE OF NOTATION ! ! $y = y(x)$!

In this chapter we will denote the variable in y by x rather than by t. This will be useful for the later application.

We study the ODE on some interval $[x_1, x_2]$ = set of x such that $x_1 \le x \le x_2$



where $x_1 < x_2$ are given numbers.

We pose the <u>boundary value problem</u>

$$\begin{cases} P(D)y = 0 \\ y(x_1) = Y_1 \\ y(x_2) = Y_2 \end{cases}$$

where Y_1, Y_2 are given numbers

We call $y(x_1) = Y_1$, $y(x_2) = Y_2$ <u>boundary conditions</u>.

If $Y_1 = Y_2 = 0$ then the boundary conditions are called <u>homogeneous</u>.

Here is a quick comparison:

INITIAL CONDITIONS	BOUNDARY CONDITIONS
$y(x_1) = A_0$, $y'(x_1) = A_1$	$y(x_1) = Y_1$, $y(x_2) = Y_2$
y & y' at <u>same pt</u>	y at <u>2 different points</u>
Solution always exists & is unique	Solution might not exist or be unique !!

ALGORITHM for solving a boundary value problem (BVP)

$\boxed{\text{TECHNIQUE}}$

Step 1: Find the general solution of the differential equation. It depends on 2 constants C_1, C_2: $y(x) = C_1 y_1(x) + C_2 y_2(x)$

Step 2: Plug in the initial condition to get a system of 2 linear equations on C_1, C_2

$$\begin{cases} C_1 \cdot y_1(x_1) + C_2 \cdot y_2(x_1) = Y_1 \\ C_1 \cdot y_1(x_2) + C_2 \cdot y_2(x_2) = Y_2 \end{cases}$$

Solve it to find C_1, C_2.

Note: the above system of linear equations has the form

$\boxed{\text{THEORY}}$

$$\begin{pmatrix} y_1(x_1) & y_2(x_1) \\ y_1(x_2) & y_2(x_2) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

Recalling the criteria for invertibility of the matrix $W = \begin{pmatrix} y_1(x_1) & y_2(x_1) \\ y_1(x_2) & y_2(x_2) \end{pmatrix}$

We get 2 cases ($\det W = y_1(x_1) y_2(x_2) - y_1(x_2) y_2(x_1)$)

Case 1: $\det W \neq 0$. Then the boundary value problem has a unique solution for any Y_1, Y_2. In this case we say the BVP is $\boxed{\text{well-posed}}$.

Case 2: $\det W = 0$. Then:

- For some Y_1, Y_2 (e.g. $Y_1 = Y_2 = 0$) the BVP has infinitely many solutions (as the null space of W has a $\neq 0$ vector)
- For some Y_1, Y_2 the BVP has no solutions (as the column space of W is not the entire $\mathbb{R}^2$)

In this case we say the BVP (boundary value problem) is $\boxed{\text{ill-posed}}$.

THEORY

Exercise: Find all solutions to the following BVPs. Are these BVPs well-posed?

PRACTICE

(a) $\begin{cases} y'' - y = 0 \\ y(0) = 1 \\ y(1) = 1 \end{cases}$ (b) $\begin{cases} y'' + y = 0 \\ y(0) = 1 \\ y(\pi) = -1 \end{cases}$

Solution: (a) General solution $y = C_1 e^x + C_2 e^{-x}$. Plug in the boundary conditions:

$$\begin{cases} y(0) = C_1 + C_2 = 1 \\ y(1) = C_1 e + C_2 e^{-1} = 1 \end{cases} \qquad \begin{pmatrix} 1 & 1 \\ e & e^{-1} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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$$W = \begin{pmatrix} 1 & 1 \\ e & e^{-1} \end{pmatrix} \to \det W = e^{-1} - e \neq 0.$$

The BVP is $\boxed{\text{well-posed}}$.

$$W^{-1} = \frac{1}{\det W} \begin{pmatrix} e^{-1} & -1 \\ -e & 1 \end{pmatrix}$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = W^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\det W} \begin{pmatrix} e^{-1} & -1 \\ -e & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{e^{-1} - e} \begin{pmatrix} e^{-1} - 1 \\ 1 - e \end{pmatrix} = \frac{1}{1 - e^2} \begin{pmatrix} 1 - e \\ e - e^2 \end{pmatrix} = \frac{1}{1+e} \begin{pmatrix} 1 \\ e \end{pmatrix}$$

Get $y(x) = \dfrac{e^x + e \cdot e^{-x}}{1+e} = \dfrac{e^x + e^{1-x}}{1+e}$

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b **General solution**: $y(x) = C_1 \cos x + C_2 \sin x$

Plug in the boundary conditions:

$$\begin{cases} y(0) = C_1 = 1 \\ y(\pi) = -C_1 = -1 \end{cases} \qquad \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\det \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = 0 \Rightarrow$ BVP is $\boxed{\text{ill-posed}}$

Solving the system we have $C_1 = 1$, C_2 any

So we get $y(x) = \cos x + C_2 \sin x$.