§8.3. Exponential of a matrix and inhomogeneous systems

§8.3.1. Matrix exponential

This is a very useful tool to write the general solution to the system $\boxed{\vec{y}' = A\vec{y}}$

The general solution will be written as

$$\boxed{\vec{y}(t) = e^{tA} \cdot \vec{w}}$$

where e^{tA} is the exponential of the matrix tA, defined below;
$\vec{w}$ is an arbitrary (constant) vector in $\mathbb{R}^n$

This is analogous to 1^{st} order ODEs $y' = \lambda y$ whose general solution is

$$y(t) = e^{t\lambda} \cdot c, \quad c \text{ any number}$$

THEORY

Definition Let A be an $n \times n$ matrix.
The exponential of A is the $n \times n$ matrix

$$e^A \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} \frac{A^j}{j!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

where $j! = 1 \cdot 2 \cdot 3 \cdots n$

<u>Remark</u>. The series $\sum_{j=0}^{\infty} \frac{A^j}{j!}$ converges.

We won't show this here, and we will only use the definition to derive the properties of the exponential (whose proofs are optional in this course)

So <u>you don't have to remember the definition.</u>

The definition is by analogy with

$$e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!} \quad \text{where } z \text{ is a number}$$

Properties of matrix exponential

(proofs are <u>optional</u>) THEORY

1. $e^0 = I$, $(e^A)^{-1} = e^{-A}$

2. If $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ is a diagonal matrix then

$$e^D = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$$ is also a diagonal matrix

<u>Proof</u> Follows from the definition of e^D

Since $D^j = \begin{pmatrix} \lambda_1^j & & 0 \\ & \ddots & \\ 0 & & \lambda_n^j \end{pmatrix}$

3. If S is an invertible matrix then

$\boxed{e^{SAS^{-1}} = Se^AS^{-1}}$ for any matrix A

Proof Use the definition

$(SAS^{-1})^j = SA^jS^{-1}$, so

$$e^{SAS^{-1}} = \sum_{j=0}^{\infty} \frac{(SAS^{-1})^j}{j!} = \sum_{j=0}^{\infty} \frac{SA^jS^{-1}}{j!}$$

$$= S \sum_{j=0}^{\infty} \frac{A^j}{j!} S^{-1} = Se^AS^{-1}. \quad \square$$

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4 $\frac{d}{dt} e^{tA} = Ae^{tA} = e^{tA} \cdot A$

Proof Write $e^{tA} = \sum_{j=0}^{\infty} \frac{t^jA^j}{j!}$

and differentiate term by term:

$$\frac{d}{dt} e^{tA} = \sum_{j=0}^{\infty} \frac{jt^{j-1}A^j}{j!} = \sum_{j=1}^{\infty} \frac{t^{j-1}A^j}{(j-1)!} = \sum_{k=0}^{\infty} \frac{t^kA^{k+1}}{k!}$$

On the other hand

$$Ae^{tA} = A\sum_{k=0}^{\infty} \frac{t^kA^k}{k!} = \sum_{k=0}^{\infty} \frac{t^kA^{k+1}}{k!}$$

and same is true for $e^{tA} \cdot A$. $\square$

Caution: for general matrices A, B

$$e^{A+B} \neq e^Ae^B \quad !$$

But we do have $\boxed{e^{(t+s)A} = e^{tA}e^{sA}}$

for any numbers t, s

18.03
§8.3
4

Properties 2 - 3 give us the following formula for computing e^A when A is diagonalizable:

if $A = SDS^{-1}$, $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

then $e^A = Se^DS^{-1}$, $e^D = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$ (*)

ALGORITHM for finding e^A when A is diagonalizable: $\boxed{\text{TECHNIQUE}}$

Step 1: find a basis of eigenvectors
$A\vec{v}_1 = \lambda_1 \vec{v}_1, \dots, A\vec{v}_n = \lambda_n \vec{v}_n$

Step 2: write $A = SDS^{-1}$ where
$S = (\vec{v}_1 \cdots \vec{v}_n)$, $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

Step 3: Compute e^A using the formula (*).

Same can be done for e^{tA} where t is any number: $tA = S \cdot tD \cdot S^{-1}$

Example 1: find e^{tA} where

PRACTICE

$$A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$$

Step 1: We studied this example in §7.3 and found $A\vec{v}_1 = \lambda_1 \vec{v}_1$, $A\vec{v}_2 = \lambda_2 \vec{v}_2$ where $\lambda_1 = 1$, $\lambda_2 = 2$, $\vec{v}_1 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Step 2: $A = SDS^{-1}$ where

$$S = \begin{pmatrix} -1/2 & -1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Compute S^{-1}: $\det S = 1/2$,

$$S^{-1} = \frac{1}{\det S} \begin{pmatrix} 1 & 1 \\ -1 & -1/2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -2 & -1 \end{pmatrix}$$

Step 3: $e^{tA} = Se^{tD}S^{-1} = S \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} S^{-1}$

$$= \begin{pmatrix} -1/2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{e^t}{2} & -e^{2t} \\ e^t & e^{2t} \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} -e^t + 2e^{2t} & -e^t + e^{2t} \\ 2e^t - 2e^{2t} & 2e^t - e^{2t} \end{pmatrix}$$

Example 2: Find e^{tA} where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Step 1: We studied this example back in §5.2 and found $A\vec{v}_1 = \lambda_1 \vec{v}_1$, $A\vec{v}_2 = \lambda_2 \vec{v}_2$ where

$$\lambda_1 = i, \ \lambda_2 = -i, \ \vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Step 2: $A = SDS^{-1}$ where

$$S = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \ D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Compute S^{-1}: $\det S = -2i$, $\frac{1}{\det S} = \frac{i}{2}$,

$$S^{-1} = \frac{1}{\det S}\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{pmatrix}$$

Step 3: $e^{tA} = Se^{tD}S^{-1} = S\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}S^{-1}$

$$= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}\begin{pmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{pmatrix}$$

$$\boxed{\begin{aligned} e^{it} &= \cos t + i \sin t \\ e^{-it} &= \cos t - i \sin t \end{aligned}}$$

$$= \begin{pmatrix} e^{it} & e^{-it} \\ ie^{it} & -ie^{-it} \end{pmatrix}\begin{pmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{pmatrix} = \begin{pmatrix} \frac{e^{it}+e^{-it}}{2} & i\frac{e^{-it}-e^{it}}{2} \\ i\frac{e^{it}-e^{-it}}{2} & \frac{e^{it}+e^{-it}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

Note: even though λ_1, λ_2 are complex, e^{tA} has real entries.

§8.3.2. Solving $\vec{y}' = A\vec{y}$ using matrix exp

From property 4 above:

$$\frac{d}{dt} e^{tA} = A e^{tA}$$

Now, if $\vec{w}$ is any vector and

THEORY

$$(*) \quad \boxed{\vec{y}(t) = e^{tA}\vec{w}}$$

then $\frac{d}{dt}\vec{y} = \left(\frac{d}{dt}e^{tA}\right)\vec{w} = Ae^{tA}\vec{w} = A\vec{y}$

That is, $\vec{y}' = A\vec{y}$.

We also have $\vec{y}(0) = e^{0}\vec{w} = I\vec{w} = \vec{w}$.

<u>Theorem</u> Let A be an $n\times n$ matrix. Then:

1. The general solution to $\vec{y}' = A\vec{y}$ has the form $(*)$ where $\vec{w}$ is an arbitrary (constant) vector
2. Fix $\vec{w}$. Then $(*)$ gives the unique solution to the initial value problem

$$\begin{cases} \vec{y}' = A\vec{y} \\ \vec{y}(0) = \vec{w} \end{cases}$$

Example: find the general solution to the initial value problem

$$\begin{cases} y_1' = y_2 \\ y_2' = -y_1 \\ y_1(0) = w_1 \\ y_2(0) = w_2 \end{cases}$$

$\boxed{\text{PRACTICE}}$
$\boxed{\text{TECHNIQUE}}$

Solution: write the system in vector form

$$\begin{cases} \vec{y}\,' = A\vec{y} \\ \vec{y}(0) = \vec{w} \end{cases} \quad \text{where } A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

The solution is $\vec{y}(t) = e^{tA}\vec{w}$.

We previously computed $e^{tA} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$

Thus $\vec{y}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$$= \begin{pmatrix} \cos t \cdot w_1 + \sin t \cdot w_2 \\ -\sin t \cdot w_1 + \cos t \cdot w_2 \end{pmatrix}$$

Remark: the formulas $\vec{y}(t) = e^{tA}\vec{w}$ and $\vec{y}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + C_n e^{\lambda_n t}\vec{v}_n$ (from §8.2) both give the general solution to $\vec{y}\,' = A\vec{y}$.
They have similarities (both need you to find the eigenvalues and eigenvectors of A)

The formula $\vec{y}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + \cdots + c_n e^{\lambda_n t} \vec{v}_n$ is a bit faster to use (no need to find S^{-1})
but the formula $\vec{y}(t) = e^{tA} \vec{w}$ is easier to use to solve the initial value problem and for inhomogeneous equations (which we study later)

For a more general way of writing the general solution, using a <u>fundamental matrix</u>, see $\boxed{\text{MITx } 3.5.7 - 3.5.8}$

<u>Interpretation</u> of the matrix exponential: if $\vec{y}(t)$ solves $\vec{y}\,' = A\vec{y}$ then for each t, s

$$\boxed{\vec{y}(t+s) = e^{tA} \vec{y}(s)}.$$

THEORY

<u>Proof</u>: $\vec{y}(t) = e^{tA} \vec{w}$ for some $\vec{w}$

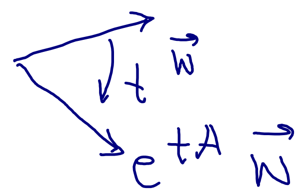
$$\vec{y}(t+s) = e^{(t+s)A} \vec{w} = e^{tA} e^{sA} \vec{w} = e^{tA} \vec{y}(s). \square$$

That is, multiplication by e^{tA} maps the state of the system at time s to the state at time $t+s$.

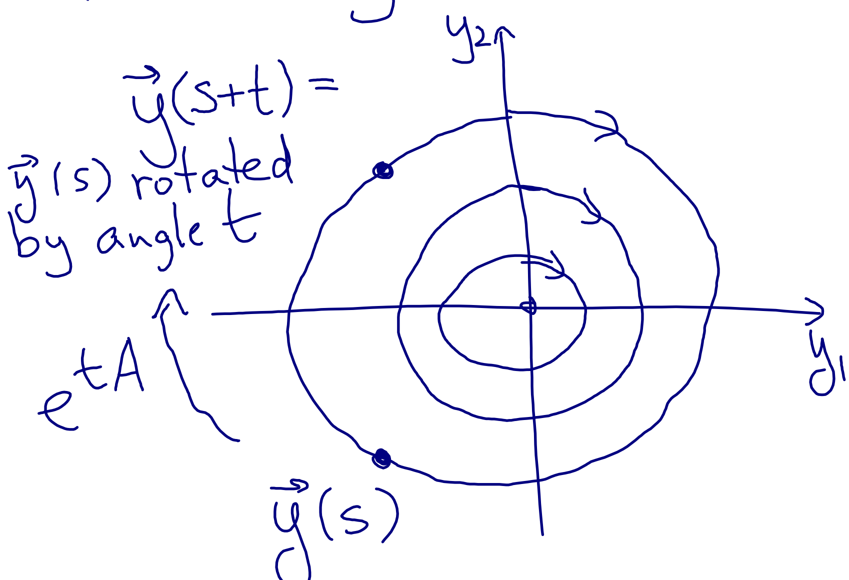
Coming back to the previous example:

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $e^{tA} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$

Multiplying by e^{tA} gives <u>clockwise rotation</u> by angle t:



This is consistent with the <u>phase portrait</u> of the system (see §6.3):



§8.3.3. Inhomogeneous systems and variation of parameters

We now study inhomogeneous systems

$\vec{y}\,'(t) = A\vec{y}(t) + \vec{b}(t)$ where $\begin{cases} A \text{ is a given } n\times n \text{ matrix} \\ \vec{b} \text{ is a given vector depending on } t \end{cases}$

We solve these for any $\vec{b}(t)$ using variation of parameters: TECHNIQUE

Step 1 Look for the solution to $\vec{y}' = A\vec{y} + \vec{b}$ in the form

$$\boxed{\vec{y}(t) = e^{tA}\vec{u}(t)}$$

i.e. take the general solution to the homogeneous equation, $\vec{y}(t) = e^{tA}\vec{w}$, and replace the constant vector $\vec{w}$ by a t-dependent vector $\vec{u}$

Step 2 Plug into the inhomogeneous system:

$$\vec{y}'(t) = A\vec{y}(t) + \vec{b}(t)$$

$$\left(e^{tA}\vec{u}(t)\right)' = Ae^{tA}\vec{u}(t) + \vec{b}(t)$$

$$\boxed{\text{Use } \frac{d}{dt}e^{tA} = Ae^{tA}:}$$

$$\underbrace{Ae^{tA}\vec{u}(t)} + e^{tA}\vec{u}'(t) = \underbrace{Ae^{tA}\vec{u}(t)} + \vec{b}(t)$$

cancellation

Get $e^{tA}\vec{u}'(t) = \vec{b}(t)$, so

$$\boxed{\vec{u}'(t) = e^{-tA}\vec{b}(t)}$$

<u>Step 3</u>: integrate to get

$\vec{u}(t) = \int e^{-tA}\vec{b}(t)\,dt \; (+\vec{c}$, any constant vector)

Then $\vec{y}(t) = e^{tA}\vec{u}(t)$ is the general solution.

<u>Example</u>: find the general solution to

$$y'' + y = \cos t.$$

PRACTICE
TECHNIQUE

<u>Solution</u>: <u>Step 0</u>: write in vector form using companion system

$\vec{y} = \begin{pmatrix} y \\ y' \end{pmatrix}$, $\begin{cases} y' = y' \\ y'' = -y + \cos t \end{cases}$

$\vec{y}\,' = A\vec{y} + \vec{b}(t)$ where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\vec{b}(t) = \begin{pmatrix} 0 \\ \cos t \end{pmatrix}$

<u>Step 1</u>: Looking in the form $\vec{y}(t) = e^{tA}\vec{u}(t)$

We previously computed $e^{tA} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$

<u>Step 2</u>: get $\vec{u}\,'(t) = e^{-tA}\vec{b}(t)$

Compute $e^{-tA}\vec{b}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} 0 \\ \cos t \end{pmatrix} = \begin{pmatrix} -\sin t \cos t \\ \cos^2 t \end{pmatrix}$

Step 3: integrate

$$\vec{u}'(t) = \begin{pmatrix} -\sin t \cos t \\ \cos^2 t \end{pmatrix}, \text{ i.e.}$$

$$\begin{cases} u_1'(t) = -\sin t \cos t \\ u_2'(t) = \cos^2 t \end{cases}$$

$$u_1(t) = \int -\sin t \cos t \, dt = -\frac{1}{2}\int \sin(2t)\,dt$$
$$= \frac{1}{4}\cos(2t) + C_1$$

$$u_2(t) = \int \cos^2 t \, dt = \int \frac{1+\cos(2t)}{2}\,dt$$
$$= \frac{t}{2} + \frac{1}{4}\sin(2t) + C_2$$

Now $\vec{y}(t) = e^{tA}\vec{u}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} \frac{1}{4}\cos(2t) + C_1 \\ \frac{t}{2} + \frac{1}{4}\sin(2t) + C_2 \end{pmatrix}$

We only need the first entry which is

$$y(t) = \frac{1}{4}\cos t \cos(2t) + C_1 \cos t + \frac{t}{2}\sin t + \frac{1}{4}\sin t \sin(2t) + C_2 \sin t$$

$$= \frac{1}{4}(\underbrace{\cos^3 t - \cos t \cdot \sin^2 t + 2\cos t \cdot \sin^2 t}_{=\cos t}) + \frac{t}{2}\sin t + C_1 \cos t + C_2 \sin t$$

$$= \boxed{\frac{t}{2}\sin t + \left(C_1 + \frac{1}{4}\right)\cos t + C_2 \sin t.}$$