§8.3. Exponential of a matrix and inhomogeneous systems

§8.3.1. Matrix exponential

This is a very useful tool to write the general solution to the system $\vec{y}' = A\vec{y}$

The general solution will be written as

$$\vec{y}(t) = e^{tA} \cdot \vec{w}$$

where e^{tA} is the exponential of the matrix tA, defined below;
$\vec{w}$ is an arbitrary (constant) vector in $\mathbb{R}^n$

This is analogous to 1^{st} order ODEs $y' = \lambda y$ whose general solution is

$$y(t) = e^{t\lambda} \cdot c, \quad c \text{ any number}$$

THEORY

Definition Let A be an $n \times n$ matrix.
The exponential of A is the $n \times n$ matrix

$$e^A \overset{def}{=} \sum_{j=0}^{\infty} \frac{A^j}{j!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

where $j! = 1 \cdot 2 \cdot 3 \cdots n$

Remark. The series $\sum_{j=0}^{\infty} \frac{A^j}{j!}$ converges.

We won't show this here, and we will only use the definition to derive the properties of the exponential (whose proofs are optional in this course)

So you don't have to remember the definition.

The definition is by analogy with

$$e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!} \text{ where } z \text{ is a number}$$

Properties of matrix exponential

(proofs are optional) THEORY

1. $e^0 = I$, $(e^A)^{-1} = e^{-A}$

2. If $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ is a diagonal matrix then

$$e^D = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$$ is also a diagonal matrix

Proof Follows from the definition of e^D

Since $D^j = \begin{pmatrix} \lambda_1^j & & 0 \\ & \ddots & \\ 0 & & \lambda_n^j \end{pmatrix}$

3. If S is an invertible matrix then

$$\boxed{e^{SAS^{-1}} = Se^AS^{-1}}$$ for any matrix A

Proof Use the definition $(SAS^{-1})^j = SA^jS^{-1}$, so

$$e^{SAS^{-1}} = \sum_{j=0}^{\infty} \frac{(SAS^{-1})^j}{j!} = \sum_{j=0}^{\infty} \frac{SA^jS^{-1}}{j!}$$

$$= S \sum_{j=0}^{\infty} \frac{A^j}{j!} S^{-1} = Se^A S^{-1}. \quad \square$$

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(4) $\frac{d}{dt} e^{tA} = Ae^{tA} = e^{tA} \cdot A$

Proof Write $e^{tA} = \sum_{j=0}^{\infty} \frac{t^j A^j}{j!}$ and differentiate term by term:

$$\frac{d}{dt} e^{tA} = \sum_{j=0}^{\infty} \frac{j t^{j-1} A^j}{j!} = \sum_{j=1}^{\infty} \frac{t^{j-1} A^j}{(j-1)!} = \sum_{k=0}^{\infty} \frac{t^k A^{k+1}}{k!}$$

On the other hand

$$Ae^{tA} = A \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k A^{k+1}}{k!}$$

and same is true for $e^{tA} \cdot A$. $\square$

Caution: for general matrices A, B

$$e^{A+B} \neq e^A e^B \quad !$$

But we do have $\boxed{e^{(t+s)A} = e^{tA} e^{sA}}$ for any numbers t, s

Properties (2)-(3) give us the following formula for computing e^A when A is <u>diagonalizable</u>:

if $A = SDS^{-1}$, $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

then $e^A = Se^DS^{-1}$, $e^D = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$ (*)

ALGORITHM for finding e^A when A is diagonalizable: TECHNIQUE

Step 1: find a basis of eigenvectors

$A\vec{v}_1 = \lambda_1\vec{v}_1, \ldots, A\vec{v}_n = \lambda_n\vec{v}_n$

Step 2: write $A = SDS^{-1}$ where

$S = (\vec{v}_1 \cdots \vec{v}_n)$, $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

Step 3: Compute e^A using the formula (*).

Same can be done for e^{tA} where t is any number: $tA = S \cdot tD \cdot S^{-1}$

Example 1: find e^{tA} where

$$A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$$

PRACTICE

Step 1: we studied this example in §7.3 and found $A\vec{v}_1 = \lambda_1 \vec{v}_1$, $A\vec{v}_2 = \lambda_2 \vec{v}_2$ where $\lambda_1 = 1$, $\lambda_2 = 2$, $\vec{v}_1 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Step 2: $A = SDS^{-1}$ where

$$S = \begin{pmatrix} -1/2 & -1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Compute S^{-1}: $\det S = 1/2$,

$$S^{-1} = \frac{1}{\det S} \begin{pmatrix} 1 & 1 \\ -1 & -1/2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -2 & -1 \end{pmatrix}$$

Step 3: $e^{tA} = Se^{tD}S^{-1} = S\begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} S^{-1}$

$$= \begin{pmatrix} -1/2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{e^t}{2} & -e^{2t} \\ e^t & e^{2t} \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} -e^t + 2e^{2t} & -e^t + e^{2t} \\ 2e^t - 2e^{2t} & 2e^t - e^{2t} \end{pmatrix}$$

Example 2: find e^{tA} where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Step 1: we studied this example back in §5.2 and found $A\vec{v}_1 = \lambda_1 \vec{v}_1$, $A\vec{v}_2 = \lambda_2 \vec{v}_2$ where

$\lambda_1 = i$, $\lambda_2 = -i$, $\vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Step 2: $A = SDS^{-1}$ where

$S = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$, $D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

Compute S^{-1}: $\det S = -2i$, $\frac{1}{\det S} = \frac{i}{2}$,

$$S^{-1} = \frac{1}{\det S} \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{pmatrix}$$

Step 3: $e^{tA} = Se^{tD}S^{-1} = S \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} S^{-1}$

$$= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$