

## §8.2. Solving $n \times n$ systems

18.03  
§8.2  
①

To solve an  $n \times n$  system of ODEs we can follow an algorithm similar to the one in §6.2, assuming that the matrix  $A$  is diagonalizable:

Step 1: write the system in the form

$$\vec{y}' = A\vec{y}$$

TECHNIQUE

Step 2: find a basis of eigenvectors of  $A$ :

$$A\vec{v}_1 = \lambda_1\vec{v}_1, A\vec{v}_2 = \lambda_2\vec{v}_2, \dots, A\vec{v}_n = \lambda_n\vec{v}_n$$

$\vec{v}_1, \dots, \vec{v}_n$  basis of  $\mathbb{R}^n$  (or  $\mathbb{C}^n$ )

Step 3: write the general solution as

$$\vec{y} = C_1 e^{\lambda_1 t} \vec{v}_1 + \dots + C_n e^{\lambda_n t} \vec{v}_n$$

Step 4: for any conjugate pair of complex

eigenvalues  $\lambda_j, \lambda_k$  ( $\lambda_k = \bar{\lambda}_j$ ),

replace  $e^{\lambda_j t} \vec{v}_j, e^{\lambda_k t} \vec{v}_k$  above by

$$\text{Re}(e^{\lambda_j t} \vec{v}_j), \text{Im}(e^{\lambda_j t} \vec{v}_j)$$

to get the general real solution

(i.e. just keep one of the two roots but take Re, Im parts)

# Example 1: SEIR model applied to COVID-19

18.03  
§8.2  
(2)

$$\vec{y}' = A\vec{y} \text{ where } A = \begin{pmatrix} -0.2 & 1.75 & 0 \\ 0.2 & -0.5 & 0 \\ 0 & 0.5 & 0 \end{pmatrix}$$

I cheated & asked a computer to find the eigenvalues. They were

$$\lambda_1 \approx -0.9603, \lambda_2 = 0, \lambda_3 \approx 0.2603$$

(One can also compute the eigenvectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ )

The general solution is

$$\vec{y}(t) = C_1 e^{\lambda_1 t} \vec{v}_1 + C_2 e^{\lambda_2 t} \vec{v}_2 + C_3 e^{\lambda_3 t} \vec{v}_3$$

As  $t \rightarrow \infty$ , the term  $e^{\lambda_3 t}$  dominates and our model (which is not very accurate!!!) would say that the number of infected people will be multiplied by  $e^{\lambda_3} \approx 1.2974$  every day, i.e. it will increase by  $\approx 29.7\%$  a day.



Note that  $A$  can be written in block form  $A = \begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$

Where  $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $B = \begin{pmatrix} -2\omega^2 & \omega^2 \\ \omega^2 & -2\omega^2 \end{pmatrix}$

Assume that  $\vec{w}$  is an eigenvector of  $A$ :

$A\vec{w} = \lambda\vec{w}$  for some  $\lambda$ .

Write  $\vec{w} = \begin{pmatrix} \vec{x} \\ \vec{v} \end{pmatrix}$  where  $\vec{x}, \vec{v}$  in  $\mathbb{R}^2$ .

Then  $A\vec{w} = \begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{v} \end{pmatrix} = \begin{pmatrix} \vec{v} \\ B\vec{x} \end{pmatrix}$ .

Thus  $A\vec{w} = \lambda\vec{w} \Leftrightarrow \begin{pmatrix} \vec{v} \\ B\vec{x} \end{pmatrix} = \begin{pmatrix} \lambda\vec{x} \\ \lambda\vec{v} \end{pmatrix} = \begin{cases} \vec{v} = \lambda\vec{x} \\ B\vec{x} = \lambda\vec{v} \end{cases}$

Substituting the first eqn. into the second one we get  $B\vec{x} = \lambda^2\vec{x}$

That is,  $\lambda^2$  has to be an eigenvalue of  $B$

We find eigenvalues & eigenvectors of  $B$ :

$\text{tr } B = -4\omega^2$ ,  $\det B = 3\omega^2$

$P(z) = z^2 + 4\omega^2 z + 3\omega^2 = (z + \omega^2)(z + 3\omega^2)$

Eigenvalues of  $B$ :  $z = -\omega^2, -3\omega^2$

Eigenvector of B for  $z = -\omega^2$ :

in the null space of  $B + \omega^2 I = \begin{pmatrix} -\omega^2 & \omega^2 \\ \omega^2 & -\omega^2 \end{pmatrix}$

Get  $-\omega^2 x_1 + \omega^2 x_2 = 0 \Rightarrow x_1 = x_2$ .

eigenvector:  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Eigenvector of B for  $z = -3\omega^2$ :

in the null space of  $B + 3\omega^2 I = \begin{pmatrix} \omega^2 & \omega^2 \\ \omega^2 & \omega^2 \end{pmatrix}$

Get  $\omega^2 x_1 + \omega^2 x_2 = 0 \Rightarrow x_2 = -x_1$

eigenvector:  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Now remember that

$\lambda$  eigenvalue of A  $\Rightarrow z = \lambda^2$  eigenvalue of B  
 $\Rightarrow z = -\omega^2$  or  $z = -3\omega^2$

We get  $\lambda^2 = -\omega^2$  or  $\lambda^2 = -3\omega^2$

This gives us the 4 eigenvalues of A:

$\lambda_1 = i\omega, \lambda_2 = -i\omega, \lambda_3 = \sqrt{3}i\omega, \lambda_4 = -\sqrt{3}i\omega$

Now let us find the eigenvectors of A.

We know: if  $\vec{w} = \begin{pmatrix} \vec{x} \\ \vec{v} \end{pmatrix}$  is an eigenvector

of A with eigenvalue  $\lambda$  then  $\vec{x}$  is an eigenvector of B with eigenvalue  $\lambda^2$  and  $\vec{v} = \lambda \vec{x}$

$$\lambda_1 = i\omega \rightarrow B\vec{x} = -\omega^2\vec{x}, \vec{v} = i\omega\vec{x}$$

18.03  
§ 8.2

⑥

Can take  $\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{v} = \begin{pmatrix} i\omega \\ i\omega \end{pmatrix}$

Get the first eigenvector of A:  $\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \\ i\omega \\ i\omega \end{pmatrix}$

$$\lambda_2 = -i\omega \rightarrow B\vec{x} = -\omega^2\vec{x}, \vec{v} = -i\omega\vec{x}$$

Get  $\vec{w}_2 = \begin{pmatrix} 1 \\ 1 \\ -i\omega \\ -i\omega \end{pmatrix}$

$$\lambda_3 = \sqrt{3}i\omega \rightarrow B\vec{x} = -3\omega^2\vec{x}, \vec{v} = \sqrt{3}i\omega\vec{x}$$

Get  $\vec{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \vec{w}_3 = \begin{pmatrix} 1 \\ -1 \\ \sqrt{3}i\omega \\ -\sqrt{3}i\omega \end{pmatrix}$



