§8.2. Solving $n \times n$ systems

To solve an $n\times n$ system of ODEs we can follow an algorithm similar to the one in §6.2, assuming that the matrix A is <u>diagonalizable</u>:

<u>Step 1</u>: write the system in the form

$$\vec{y}' = A\vec{y}$$

TECHNIQUE

<u>Step 2</u>: find a basis of eigenvectors of A:

$$\boxed{A\vec{v}_1 = \lambda_1 \vec{v}_1,\ A\vec{v}_2 = \lambda_2 \vec{v}_2, \dots,\ A\vec{v}_n = \lambda_n \vec{v}_n}$$

$\vec{v}_1, \dots, \vec{v}_n$ basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$)

<u>Step 3</u>: write the general solution as

$$\boxed{\vec{y} = C_1 e^{\lambda_1 t}\vec{v}_1 + \dots + C_n e^{\lambda_n t}\vec{v}_n}$$

<u>Step 4</u>: for any conjugate pair of complex eigenvalues λ_j, λ_k $(\lambda_k = \bar{\lambda}_j)$, replace $e^{\lambda_j t}\vec{v}_j, e^{\lambda_k t}\vec{v}_k$ above by

$$\boxed{\mathrm{Re}\,(e^{\lambda_j t}\vec{v}_j),\ \mathrm{Im}\,(e^{\lambda_j t}\vec{v}_j)}$$

to get the general real solution (i.e. just keep <u>one</u> of the two roots but take Re, Im parts)

Example 1: SEIR model applied to COVID-19

$\vec{y}' = A\vec{y}$ where $A = \begin{pmatrix} -0.2 & 1.75 & 0 \\ 0.2 & -0.5 & 0 \\ 0 & 0.5 & 0 \end{pmatrix}$

I cheated & asked a computer to find the eigenvalues. They were

$\lambda_1 \approx -0.9603$, $\lambda_2 = 0$, $\lambda_3 \approx 0.2603$

(One can also compute the eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$)

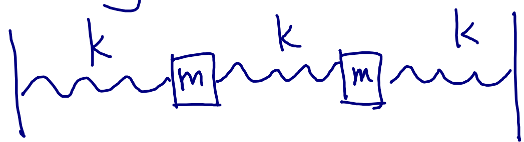
The general solution is

$\vec{y}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2 + C_3 e^{\lambda_3 t}\vec{v}_3$

As $t \to \infty$, the term $e^{\lambda_3 t}$ dominates and our model (which is not very accurate!!!) would say that the number of infected people will be multiplied by $e^{\lambda_3} \approx 1.2974$ every day, i.e. it will increase by $\approx 29.7\%$ a day.

Example 2: 3 coupled springs

where we assume that the masses & the spring constants are all the same:



$m_1 = m_2 = m$
$k_1 = k_2 = k_3 = k$

Recall that we got $\vec{y}' = A\vec{y}$

where $\vec{y} = \begin{pmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{pmatrix}$ } positions } velocities

and $$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{2k}{m} & \frac{k}{m} & 0 & 0 \\ \frac{k}{m} & -\frac{2k}{m} & 0 & 0 \end{pmatrix}$$

Define $\omega = \sqrt{\frac{k}{m}}$ (frequency of 1 spring-mass system)

then $$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2\omega^2 & \omega^2 & 0 & 0 \\ \omega^2 & -2\omega^2 & 0 & 0 \end{pmatrix}$$

To compute the eigenvalues & eigenvectors of A we could ask a computer again or we could compute $\det(\lambda I - A)$ directly. Instead we will use a trick.

Note that A can be written in block form $A = \begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -2\omega^2 & \omega^2 \\ \omega^2 & -2\omega^2 \end{pmatrix}$

Assume that $\vec{w}$ is an eigenvector of A:

$\boxed{A\vec{w} = \lambda \vec{w}}$ for some λ.

Write $\vec{w} = \begin{pmatrix} \vec{x} \\ \vec{v} \end{pmatrix}$ where $\vec{x}, \vec{v}$ in $\mathbb{R}^2$.

Then $A\vec{w} = \begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{v} \end{pmatrix} = \begin{pmatrix} \vec{v} \\ B\vec{x} \end{pmatrix}$.

Thus $A\vec{w} = \lambda\vec{w} \iff \begin{pmatrix} \vec{v} \\ B\vec{x} \end{pmatrix} = \begin{pmatrix} \lambda\vec{x} \\ \lambda\vec{v} \end{pmatrix} \iff \begin{cases} \vec{v} = \lambda\vec{x} \\ B\vec{x} = \lambda\vec{v} \end{cases}$

Substituting the first eqn. into the second one we get $\boxed{B\vec{x} = \lambda^2 \vec{x}}$

That is, λ^2 has to be an eigenvalue of B

We find eigenvalues & eigenvectors of B:

$\operatorname{tr} B = -4\omega^2$, $\det B = 3\omega^2$

$P(z) = z^2 + 4\omega^2 z + 3\omega^2 = (z + \omega^2)(z + 3\omega^2)$

Eigenvalues of B: $z = -\omega^2, -3\omega^2$

<u>Eigenvector of B for $z = -\omega^2$:</u>

in the null space of $B + \omega^2 I = \begin{pmatrix} -\omega^2 & \omega^2 \\ \omega^2 & -\omega^2 \end{pmatrix}$

Get $-\omega^2 x_1 + \omega^2 x_2 = 0 \Rightarrow x_1 = x_2$.

eigenvector: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

<u>Eigenvector of B for $z = -3\omega^2$:</u>

in the null space of $B + 3\omega^2 I = \begin{pmatrix} \omega^2 & \omega^2 \\ \omega^2 & \omega^2 \end{pmatrix}$

Get $\omega^2 x_1 + \omega^2 x_2 = 0 \Rightarrow x_2 = -x_1$

eigenvector: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Now remember that

λ eigenvalue of $A \Rightarrow z = \lambda^2$ eigenvalue of B

$\Rightarrow z = -\omega^2$ or $z = -3\omega^2$

We get $\lambda^2 = -\omega^2$ or $\lambda^2 = -3\omega^2$

This gives us the 4 eigenvalues of A:

$$\boxed{\lambda_1 = i\omega, \quad \lambda_2 = -i\omega, \quad \lambda_3 = \sqrt{3}\, i\omega, \quad \lambda_4 = -\sqrt{3}\, i\omega}$$

Now let us find the <u>eigenvectors of A</u>.

We know: if $\vec{w} = \begin{pmatrix} \vec{x} \\ \vec{v} \end{pmatrix}$ is an eigenvector of A with eigenvalue λ then $\vec{x}$ is an eigenvector of B with eigenvalue λ^2 and $\vec{v} = \lambda \vec{x}$

$\underline{\lambda_1 = i\omega}$ $\to$ $B\vec{x} = -\omega^2 \vec{x}$, $\vec{v} = i\omega\vec{x}$

Can take $\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} i\omega \\ i\omega \end{pmatrix}$

Get the first eigenvector of A: $\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \\ i\omega \\ i\omega \end{pmatrix}$

$\underline{\lambda_2 = -i\omega}$ $\to$ $B\vec{x} = -\omega^2\vec{x}$, $\vec{v} = -i\omega\vec{x}$

Get $\vec{w}_2 = \begin{pmatrix} 1 \\ 1 \\ -i\omega \\ -i\omega \end{pmatrix}$

$\underline{\lambda_3 = \sqrt{3}\, i\omega}$ $\to$ $B\vec{x} = -3\omega^2\vec{x}$, $\vec{v} = \sqrt{3}\, i\omega\vec{x}$

Get $\vec{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\Rightarrow$ $\vec{w}_3 = \begin{pmatrix} 1 \\ -1 \\ \sqrt{3}\, i\omega \\ -\sqrt{3}\, i\omega \end{pmatrix}$

$\underline{\lambda_4 = -\sqrt{3}\, i\omega}$ $\to$ $B\vec{x} = -3\omega^2\vec{x}$, $\vec{v} = -\sqrt{3}\, i\omega\vec{x}$

Get $\vec{w}_4 = \begin{pmatrix} 1 \\ -1 \\ -\sqrt{3}\, i\omega \\ \sqrt{3}\, i\omega \end{pmatrix}$

<u>General complex solution</u> to $\vec{y}' = A\vec{y}$:

$$\vec{y} = C_1 e^{i\omega t}\vec{w}_1 + C_2 e^{-i\omega t}\vec{w}_2 + C_3 e^{i\sqrt{3}\omega t}\vec{w}_3 + C_4 e^{-i\sqrt{3}\omega t}\vec{w}_4$$

C_1, C_2, C_3, C_4 arbitrary constants

The <u>general real solution</u> is

$$\boxed{\vec{y}(t) = C_1 \operatorname{Re}(e^{i\omega t}\vec{w}_1) + C_2 \operatorname{Im}(e^{i\omega t}\vec{w}_1) + C_3 \operatorname{Re}(e^{i\sqrt{3}\omega t}\vec{w}_3) + C_4 \operatorname{Im}(e^{i\sqrt{3}\omega t}\vec{w}_3)}$$

We finally compute

$$\operatorname{Re}(e^{i\omega t}\vec{w}_1) = \operatorname{Re}\left((\cos(\omega t) + i\sin(\omega t))\begin{pmatrix}1\\1\\i\omega\\i\omega\end{pmatrix}\right)$$

$$= \begin{pmatrix}\cos(\omega t)\\ \cos(\omega t)\\ -\omega\sin(\omega t)\\ -\omega\sin(\omega t)\end{pmatrix}, \quad \operatorname{Im}(e^{i\omega t}\vec{w}_1) = \begin{pmatrix}\sin(\omega t)\\ \sin(\omega t)\\ \omega\cos(\omega t)\\ \omega\cos(\omega t)\end{pmatrix}$$

$$\operatorname{Re}(e^{i\sqrt{3}\omega t}\vec{w}_3) = \operatorname{Re}\left((\cos(\sqrt{3}\omega t) + i\sin(\sqrt{3}\omega t))\begin{pmatrix}1\\-1\\ \sqrt{3}i\omega\\ -\sqrt{3}i\omega\end{pmatrix}\right)$$

$$= \begin{pmatrix}\cos(\sqrt{3}\omega t)\\ -\cos(\sqrt{3}\omega t)\\ -\sqrt{3}\omega\sin(\sqrt{3}\omega t)\\ \sqrt{3}\omega\sin(\sqrt{3}\omega t)\end{pmatrix}, \quad \operatorname{Im}(e^{i\sqrt{3}\omega t}\vec{w}_3) = \begin{pmatrix}\sin(\sqrt{3}\omega t)\\ -\sin(\sqrt{3}\omega t)\\ \sqrt{3}\omega\cos(\sqrt{3}\omega t)\\ -\sqrt{3}\omega\cos(\sqrt{3}\omega t)\end{pmatrix}$$

Get 2 frequencies of oscillation:

ω for which the masses move in the same direction

$\sqrt{3}\omega$ for which the masses move in the opposite direction: