§8. $n\times n$ systems of linear ODEs

§8.1. Basics and examples

In this chapter we study $n\times n$ syste of ODEs.
We write them in vector form:

$$\boxed{\vec{y}\,' = A\vec{y}} \qquad \text{where } \vec{y}(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$$

is the unknown vector and A is a given $n\times n$ matrix.

In §6 we studied the special case $n=2$

§8.1.1. Example: linear SEIR model

The SEIR model is a nonlinear system of 4 ODEs which can be used to model the spread of an infectious disease.
(As every model, it has its limitations.
For instance the version here assumes that nobody dies!)

The time variable is t (in days) and the unknown functions are
$S(t), E(t), I(t), R(t)$ (dimensionless)

18.03
§8.1
2

$$\left.\begin{aligned} S(t) &= \text{Susceptible}\\ E(t) &= \text{Exposed}\\ I(t) &= \text{Infected}\\ R(t) &= \text{Recovered}\end{aligned}\right\}\ \text{Population fraction at time } t$$

The equations are:

$$\begin{cases} S' = -\beta S\cdot I & (1)\\ E' = \beta S\cdot I - \alpha E & (2)\\ I' = \alpha E - \gamma I & (3)\\ R' = \gamma I & (4)\end{cases}$$

- The term $\beta S\cdot I$ in (1), (2) accounts for susceptible people becoming exposed via contact with an infected person
- The term αE in (2), (3) accounts for exposed people becoming infected (used because of incubation period)
- The term γI in (3), (4) accounts for infected people recovering

$$S \xrightarrow{\beta} E \xrightarrow{\alpha} I \xrightarrow{\gamma} R$$

18.03
5.8.1
3

The constant parameters are

α (in $\frac{1}{\text{day}}$): inverse of the incubation period

β (in $\frac{1}{\text{day}}$): average contact rate in the population (social distancing $\to$ smaller β)

γ (in $\frac{1}{\text{day}}$): inverse of the mean infectious period

For COVID-19 by one estimate:

$$\boxed{\alpha = 0.2, \quad \beta = 1.75, \quad \gamma = 0.5}$$

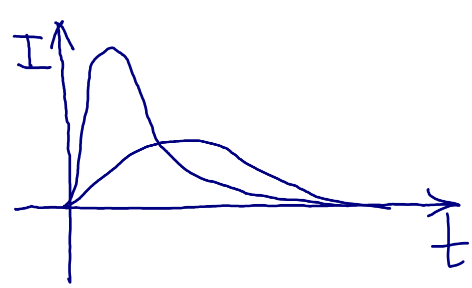
The system is nonlinear because of the term $\beta S \cdot I$ in (1) and (2).

We simplify it to a linear system by removing (1) and replacing S by 1:

$$\begin{cases} E' = \beta I - \alpha E \\ I' = \alpha E - \gamma I \\ R' = \gamma I \end{cases}$$

This might be a good approximation when E, I, R are small and $S \approx 1$ (in the initial stage).

It does not work when a large percentage of the population is infected and will not give the curves you might have seen.



Now let us write this system in vector notation: $\vec{y}(t) = \begin{pmatrix} E(t) \\ I(t) \\ R(t) \end{pmatrix}$,

$$\begin{cases} E' = -\alpha E + \beta I + 0 \cdot R \\ I' = \alpha E - \gamma I + 0 \cdot R \\ R' = 0 \cdot E + \gamma \cdot I + 0 \cdot R \end{cases}$$

Get $\vec{y}\,' = A\vec{y}$ with $A = \begin{pmatrix} -\alpha & \beta & 0 \\ \alpha & -\gamma & 0 \\ 0 & \gamma & 0 \end{pmatrix}$

For COVID-19, $A = \begin{pmatrix} -0.2 & 1.75 & 0 \\ 0.2 & -0.5 & 0 \\ 0 & 0.5 & 0 \end{pmatrix}$

---

§8.1.2. Companion systems

Similarly to 2nd order ODEs, if we have an nth order ODE

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

we can convert it to the $n \times n$ companion system

$\vec{y}\,' = A\vec{y}$ where $\vec{y} = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{pmatrix}$

Example: Find a companion system for $y''' + 2y'' + 3y' + 4y = 0$

TECHNIQUE
PRACTICE

Solution: write $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ where

$y_1 = y$
$y_2 = y'$
$y_3 = y''$

Then

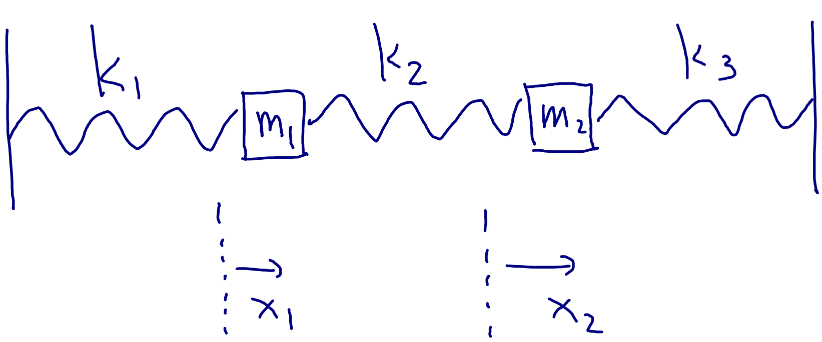
$$y_1' = y_2$$
$$y_2' = y_3$$
$$y_3' = y''' = -4y - 3y' - 2y'' = -4y_1 - 3y_2 - 2y_3$$

So $\vec{y}' = A\vec{y}$ where

$$A = \begin{array}{c} \\ y_1' \\ y_2' \\ y_3' \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{pmatrix} \end{array}$$

One can also write 1st order companion systems for systems of higher order ODEs:

Example: 3 coupled springs



$k_1, k_2, k_3 > 0$
spring constants
$m_1, m_2 > 0$
masses

We assume that there is a position of the masses in which all springs are in equilibrium and denote by the displacements of the masses from that position.

Using Newton's IInd law + Hooke's law we arrive to the 2^{nd} order syste