§7.3. Determinants, eigenvalues, and eigenvectors

§7.3.1. Determinant

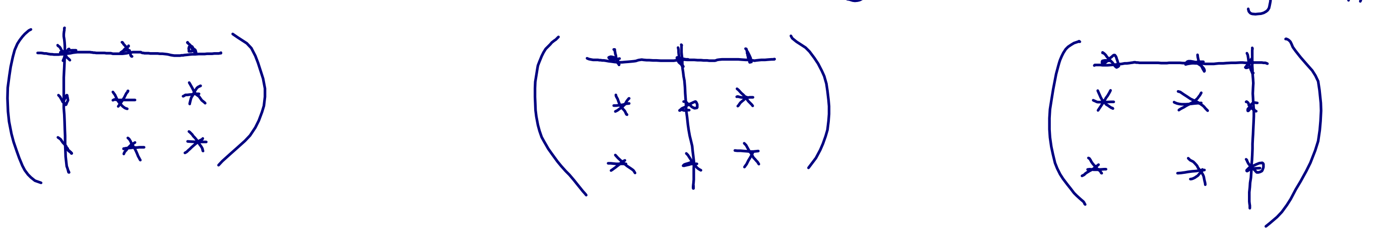
Each $n \times n$ matrix A has a number associated to it called its determinant and denoted $\boxed{\det A}$

I will not give a formal definition of $\det A$ (not useful to us here) but I will show how to compute it.

2x2 matrices: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

3x3 matrices: $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} =$

$$= a \cdot \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \cdot \det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \cdot \det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$



Example: $\det\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$= 0 \cdot \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - 1 \cdot \det\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + 1 \cdot \det\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$= 1 + 1 = 2$

Properties of det:

1. If A is upper triangular (all elements below diagonal are $=0$) then $\det A =$ product of elements on the diagonal.

Example: $\det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} = 1 \cdot 4 \cdot 6 = 24.$

2. $\det(A \cdot B) = \det A \cdot \det B$, $\det I = 1$
3. Multiplying a row of A by a number c multiplies the determinant of A by c.
4. Swapping two rows in A multiplies $\det A$ by -1.
5. Adding to a row of A a multiple of another row of A does not change $\det A$.

Using 1, 3, 4, 5 can compute $\det A$: $\boxed{\text{TECHNIQUE}}$
bring A to REF via row operations & compute the det of this REF using 1

PRACTICE

Example:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \det = -\det A$$

$$\xrightarrow{R3 \leftarrow R3 - R1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad \det = -\det A$$

$$\xrightarrow{R3 \leftarrow R3 - R2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \quad \det = -\det A$$

For the last matrix is determinant is

$$-\det A = 1 \cdot 1 \cdot -2 = -2.$$

Thus $\boxed{\det A = 2.}$

THEORY

Determinants and invertibility

Theorem. Let A be a square matrix.

Then $\boxed{A \text{ is invertible} \iff \det A \neq 0}$

§7.3.2. Eigenvalues & eigenvectors

<u>Definition</u> Let A be a square matrix. We say that a (possibly complex) number λ is an <u>eigenvalue</u> of A, if the matrix $\lambda \cdot \underset{\text{identity matrix}}{I} - A$ is not invertible (i.e. singular).

THEORY

If λ is an eigenvalue of A then $\lambda \cdot I - A$ is singular $\Rightarrow$ the null space $NS(\lambda \cdot I - A)$ contains some nonzero vectors

- We call $NS(\lambda \cdot I - A)$ the <u>eigenspace</u> of A corresponding to λ.

It consists of solutions $\vec{x}$ to $(\lambda \cdot I - A)\vec{x} = 0$, i.e. solutions of the <u>eigen-equation</u>

$$\boxed{A\vec{x} = \lambda \vec{x}} \qquad (*)$$

- We say $\vec{x}$ is an <u>eigenvector</u> of A corresponding to λ, if $\vec{x}$ solves $(*)$ and $\vec{x} \neq \vec{0}$.

As before in §5.2, to find the eigenvalues of A we use that

λ is an eigenvalue of A

$\Updownarrow$

$\lambda I - A$ is not invertible

$\Updownarrow$

$\det(\lambda I - A) = 0$.

Define $\boxed{P(\lambda) = \det(\lambda I - A)}$, if A is an $n \times n$ matrix then P is a polynomial of degree n called the <u>characteristic polynomial</u>.

$$\boxed{\text{Eigenvalues of } A = \text{Roots of } P}$$

To find a basis of each eigenspace of A, we use the null space algorithm from §7.2.1

<u>Example</u>: Find the eigenvalues & a basis for each eigenspace of $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix}$.

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Solution: $\lambda I - A = \begin{pmatrix} \lambda-1 & -1 & -1 \\ 0 & \lambda-1 & 2 \\ 0 & 0 & \lambda-2 \end{pmatrix}$

Upper triangular $\Rightarrow P(\lambda) = \det(\lambda I - A)$
$= (\lambda-1)^2(\lambda-2)$.

Eigenvalues: $1, 1, 2$ (counted with multiplicity)

Eigenspace for $\lambda = 1$: $NS\begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{pmatrix}$

Bring to RREF: $\begin{pmatrix} 0 & \boxed{-1} & -1 \\ 0 & 0 & \boxed{2} \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow{R3 \leftarrow R3 + \frac{1}{2}R2} \begin{pmatrix} 0 & \boxed{-1} & -1 \\ 0 & 0 & \boxed{2} \\ 0 & 0 & 0 \end{pmatrix}$

$\xrightarrow[R2 \leftarrow R2/2]{R1 \leftarrow -R1} \begin{pmatrix} 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R1 \leftarrow R1 - R2} \begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array} \begin{pmatrix} 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{pmatrix}$

Get $\begin{cases} x_1 \text{ any} \\ x_2 = 0 \\ x_3 = 0 \end{cases}$

$\vec{x} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Basis of $NS(I - A)$: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\text{Dim } NS(I - A) = 1$

Eigenspace for $\lambda=2$: $NS\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

$$\begin{pmatrix} \boxed{1} & -1 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R1 \leftarrow R1+R2} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

x_3 free variable

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 + 2x_3 = 0 \end{cases} \longrightarrow \begin{cases} x_1 = -x_3 \\ x_2 = -2x_3 \\ x_3 \text{ any} \end{cases}$$

$$\vec{x} = \begin{pmatrix} -x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

Basis for $NS(2I-A)$: $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$, $\dim NS(2I-A)=1$

More properties of eigenvalues:

if A has eigenvalues $\lambda_1, \dots, \lambda_n$ (with multiplicity) then

- $\lambda_1 + \dots + \lambda_n = \operatorname{tr} A$ = sum of the entries on the diagonal of A
- $\lambda_1 \cdots \lambda_n = \det A$

§7.3.3. Diagonalizability

THEORY

Let A be an $n \times n$ matrix.

We say that A is diagonalizable, if the sum of the dimensions of its eigenspaces equals n.

If A is diagonalizable, we can put together the bases of all its eigenspaces and obtain n vectors $\boxed{\vec{v}_1, \dots, \vec{v}_n}$ which will be a basis of $\mathbb{R}^n$ consisting of eigenvectors of A:

$$\boxed{A\vec{v}_1 = \lambda_1 \vec{v}_1, \dots, A\vec{v}_n = \lambda_n \vec{v}_n}$$

where $\lambda_1, \dots, \lambda_n$ are the corresponding eigenvalues

Denote $S = (\vec{v}_1 \ \dots \ \vec{v}_n)$

I.e. the columns of S are eigenvectors of A.

Then $AS = (A\vec{v}_1 \ \dots \ A\vec{v}_n) = (\lambda_1\vec{v}_1 \ \dots \ \lambda_n\vec{v}_n)$

matrix multiplication: multiply A by each column of S

Define the diagonal matrix $D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$

Then $SD = (\lambda_1\vec{v}_1 \ \dots \ \lambda_n\vec{v}_n)$

matrix multiplication: take linear combination of columns of S with coefficients coming from D.

Thus $AS = SD$, i.e. $\boxed{A = SDS^{-1}}$

TECHNIQUE

To recap: if

- A is diagonalizable,
- $A\vec{v}_1 = \lambda_1 \vec{v}_1, \dots, A\vec{v}_n = \lambda_n \vec{v}_n$, $\vec{v}_1, \dots, \vec{v}_n$ basis
- $S = \begin{pmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{pmatrix}$
- $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix}$

then $\boxed{A = SDS^{-1}}$ ← this is called <u>diagonalizing</u> A

PRACTICE

<u>Exercises:</u> Are the following matrices diagonalizable? If so, diagonalize them

a $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix}$ b $\begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$ c $\begin{pmatrix} -1 & 0 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

<u>Solutions</u> a <u>Not diagonalizable</u>:

we found before that it has eigenvalues 1 and 2 & each of the corresponding eigenspaces has dimension 1.

$$\underbrace{1+1}_{\text{sum of dim of eigenspaces}} < 3 \leftarrow \text{size of the matrix}$$

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(b) $A=\begin{pmatrix}3 & 1\\ -2 & 0\end{pmatrix}$, $\operatorname{tr} A = 3$, $\det A = 2$

Characteristic polynomial $P(\lambda)=\lambda^2-3\lambda+2$

Eigenvalues $\lambda = \dfrac{3\pm\sqrt{3^2-4\cdot 2}}{2} = 1, 2$

$\lambda=1$ eigenspace: $NS(I-A) = NS\begin{array}{cc} x_1 & x_2 \\ \end{array}\begin{pmatrix}-2 & -1\\ 2 & 1\end{pmatrix}$

$\begin{pmatrix}\boxed{-2} & -1\\ 2 & 1\end{pmatrix} \xrightarrow{R2\leftarrow R2+R1} \begin{pmatrix}\boxed{-2} & -1\\ 0 & 0\end{pmatrix}$ $\begin{cases}-2x_1-x_2=0\\ x_2 \text{ free}\end{cases}$

$\begin{cases}x_1 = -\frac{1}{2}x_2\\ x_2 \text{ any}\end{cases}$ $\vec{x}=\begin{pmatrix}-\frac{1}{2}x_2\\ x_2\end{pmatrix} = x_2\begin{pmatrix}-\frac{1}{2}\\ 1\end{pmatrix}$

Basis: $\boxed{\vec{v}_1=\begin{pmatrix}-1/2\\ 1\end{pmatrix}}$

$\lambda=2$ eigenspace: $NS(2I-A) = NS\begin{pmatrix}-1 & -1\\ 2 & 2\end{pmatrix}$

$\begin{array}{cc} x_1 & x_2 \end{array}$

$\begin{pmatrix}\boxed{-1} & -1\\ 2 & 2\end{pmatrix} \xrightarrow{R2\leftarrow R2+2R1} \begin{pmatrix}\boxed{-1} & -1\\ 0 & 0\end{pmatrix}$ $\begin{cases}x_1+x_2=0\\ x_2 \text{ any}\end{cases}$

$\begin{cases}x_1=-x_2\\ x_2 \text{ any}\end{cases}$ $\vec{x}=\begin{pmatrix}-x_2\\ x_2\end{pmatrix} = x_2\begin{pmatrix}-1\\ 1\end{pmatrix}$

Basis: $\boxed{\vec{v}_2=\begin{pmatrix}-1\\ 1\end{pmatrix}}$

A is diagonalizable, with $A=SDS^{-1}$

where $S=\begin{pmatrix}-1/2 & -1\\ 1 & 1\end{pmatrix}$, $D=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$

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(c) $A = \begin{pmatrix} -1 & 0 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ upper triangular

$$P(\lambda) = \det \begin{pmatrix} \lambda+1 & 0 & -3 \\ 0 & \lambda+1 & -2 \\ 0 & 0 & \lambda-1 \end{pmatrix} = (\lambda+1)^2(\lambda-1)$$

Eigenvalues: $-1, -1, 1$

Eigenspace for $\lambda = -1$: $NS \begin{pmatrix} 0 & 0 & -3 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 & -3 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{pmatrix} \longrightarrow \begin{array}{ccc} & & \\ \end{array}\begin{pmatrix} 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$\quad\quad x_1 \quad x_2 \quad x_3$

$\begin{cases} x_3 = 0 \\ x_1, x_2 \text{ any} \end{cases}$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Basis: $\boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$

Eigenspace for $\lambda = 1$: $NS \begin{pmatrix} 2 & 0 & -3 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

$$\begin{pmatrix} \boxed{2} & 0 & -3 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[R2 \leftarrow \frac{R2}{2}]{R1 \leftarrow \frac{R1}{2}} \begin{pmatrix} \boxed{1} & 0 & -\frac{3}{2} \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{pmatrix}$$
$\quad\quad x_1 \quad x_2 \quad x_3$

$\begin{cases} x_1 = 3/2 \cdot x_3 \\ x_2 = x_3 \\ x_3 \text{ any} \end{cases}$

$$\vec{x} = \begin{pmatrix} \frac{3}{2} \cdot x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 3/2 \\ 1 \\ 1 \end{pmatrix}$$

Basis: $\boxed{\vec{v}_3 = \begin{pmatrix} 3/2 \\ 1 \\ 1 \end{pmatrix}}$

So A is diagonalizable and

$A = SDS^{-1}$ where

$$S = \begin{pmatrix} 1 & 0 & 3/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

As we saw in exercise (a) above, not every matrix is diagonalizable.

However, if A has simple eigenvalues (i.e. the characteristic polynomial $P(\lambda)$ has no multiple roots) then A is diagonalizable.

Here is another assumption that implies diagonalizability:

Theorem (Eigenvalues & eigenvectors for symmetric matrices)

Assume that A is a symmetric matrix, i.e. if a_{jk} are its entries then $a_{jk} = a_{kj}$

(e.g. $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ symmetric, $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ not symmetric)

Then:
1. A is diagonalizable
2. All eigenvalues of A are real
3. There exists a basis $\vec{v}_1, \dots, \vec{v}_n$ of eigenvectors of A which is orthogonal:

$\vec{v}_j \cdot \vec{v}_k = 0$ for $j \neq k$ where $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \overset{\text{def}}{=} x_1 y_1 + \cdots + x_n y_n$