§7.2. Null space, column space, rank, invertibility

Here we learn how to answer the following questions for a given matrix A:

- For which vectors $\vec{b}$ does $A\vec{x} = \vec{b}$ have a solution? (using <u>column space</u>)
- Does $A\vec{x} = \vec{b}$ have a unique solution? (using <u>null space</u>)

And related questions for a set of vectors $\vec{x}_1, \ldots, \vec{x}_m$ in $\mathbb{R}^n$:

- Are these vectors <u>linearly independent</u>? (using null space)
- What is the <u>dimension</u> of $\mathrm{span}(\vec{x}_1, \ldots, \vec{x}_m)$? (using <u>rank</u>)
- Find a <u>basis</u> for $\mathrm{span}(\vec{x}_1, \ldots, \vec{x}_m)$ (using column space)

We also study <u>invertible</u> matrices A, for which $A\vec{x} = \vec{b}$ has a <u>unique</u> solution $\vec{x} = A^{-1}\vec{b}$ for <u>every</u> $\vec{b}$

and the <u>rank-nullity theorem</u>

All of these will use Row Echelon Form + Gaussian elimination

§7.2.1. Null space

Recall from §5.3 that for an $n \times m$ matrix A the <u>null space</u> of A is the set of all solutions $\vec{x}$ in $\mathbb{R}^m$ to the equation

$$\boxed{A\vec{x} = \vec{0}}$$

We denote the null space of A by $\boxed{NS(A)}$

Recall also the

<u>Definition</u> A subset S of $\mathbb{R}^m$ is called a <u>subspace</u> of $\mathbb{R}^m$, if:

- For all $\vec{x}, \vec{y}$ in S, the vector $\vec{x}+\vec{y}$ is also in S (S is closed under addition)
- For all $\vec{x}$ in S, c in $\mathbb{R}$, the vector $c\cdot\vec{x}$ is also in S (S is closed under multiplication by scalars)

<u>Note</u>: S is a subspace of $\mathbb{R}^2$ if

(a) $S = \vec{0}$, or (b) $S =$ line through $\vec{0}$, or (c) $S = \mathbb{R}^2$

For any $n \times m$ matrix A, the set $NS(A)$ is a subspace of $\mathbb{R}^m$:

$$\begin{cases} A\vec{x} = A\vec{y} = \vec{0} \Rightarrow A(\vec{x}+\vec{y}) = \vec{0} \\ A\vec{x} = 0 \Rightarrow A(c\vec{x}) = \vec{0} \end{cases}$$

ALGORITHM for finding basis & dimension

$\boxed{\text{TECHNIQUE}}$ of the null space of a matrix A:

Step 1 Convert $(A|0)$ to REF using elimination

Note: the null space stays the same

Step 2: find the general solution to $A\vec{x} = \vec{0}$, depending on arbitrary values of the free variables

Step 3: Write the formula for the general solution as a linear combination with free variables as coefficients.

The vectors in the linear combination form a basis for the null space $NS(A)$

Note: dimension of $NS(A)$ (number of elements in its basis)
$\parallel$
# of free variables in the REF of A

$\boxed{\text{PRACTICE}}$

Example 1: $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Step 1: bring $(A|0)$ to REF

$$\left(\begin{array}{ccc|c} \boxed{2} & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{R2 \Leftarrow R2 - \frac{R1}{2}} \left(\begin{array}{ccc|c} \boxed{2} & 1 & 1 & 0 \\ 0 & \boxed{\frac{1}{2}} & -\frac{1}{2} & 0 \end{array}\right)$$

Step 2: find the general solution to $A\vec{x} = \vec{0}$: use augmented matrix $(A \mid 0)$

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} \boxed{2} & 1 & 1 & 0 \\ 0 & \boxed{\frac{1}{2}} & -\frac{1}{2} & 0 \end{array}\right)$$

x_1, x_2 dependent variables

x_3 free variable

$$\begin{cases} 2x_1 + x_2 + x_3 = 0 \\ \frac{1}{2} x_2 - \frac{1}{2} x_3 = 0 \end{cases}$$

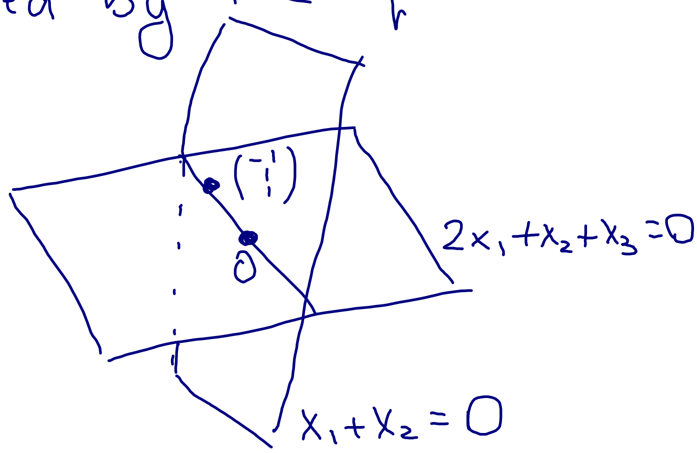
Get $x_2 = x_3$, $x_1 = -\dfrac{x_2 + x_3}{2} = -x_3$

General solution: $\begin{cases} x_1 = -x_3 \\ x_2 = x_3 \\ x_3 \text{ any} \end{cases}$

Step 3: write $\vec{x} = \begin{pmatrix} -x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

Basis for NS(A): $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, dimension (NS(A)) = 1.

Geometric picture: the intersection of two planes in $\mathbb{R}^3$ defined by the equations $2x_1 + x_2 + x_3 = 0$ and $x_1 + x_2 = 0$ is the line spanned by the vector $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$



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Example 2: $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix}$

Step 1: $\left(\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 5 & 0 \end{array}\right) \xrightarrow{R2 \leftarrow R2 - 2R1} \left(\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 4 & 0 \\ 0 & \boxed{-1} & -2 & -3 & 0 \end{array}\right)$

Let's actually convert this to RREF:

$$\left(\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 4 & 0 \\ 0 & \boxed{-1} & -2 & -3 & 0 \end{array}\right) \xrightarrow{R2 \leftarrow -R2} \left(\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 4 & 0 \\ 0 & \boxed{1} & 2 & 3 & 0 \end{array}\right)$$

$$\downarrow R1 \leftarrow R1 - 2 \cdot R2$$

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & -1 & -2 & 0 \\ 0 & \boxed{1} & 2 & 3 & 0 \end{array}\right)$$

Step 2: x_1, x_2 dependent variables

x_3, x_4 free variables

$$\begin{cases} x_1 - x_3 - 2x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{cases}$$

General solution to $A\vec{x} = \vec{0}$: $\begin{cases} x_1 = x_3 + 2x_4 \\ x_2 = -2x_3 - 3x_4 \\ x_3, x_4 \text{ any} \end{cases}$

Step 3: Write the general solution above as

$$\vec{x} = \begin{pmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

Basis for $NS(A)$: $\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$, $\dim(NS(A)) = 2$.

The null space lets us find out when a given set of vectors is <u>linearly independent</u>, using the following

<u>Fact</u>: If $A = [\vec{v}_1 \dots \vec{v}_m]$ is an $n \times m$ matrix with columns $\vec{v}_1, \dots, \vec{v}_m$, then for any vector $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$ we have

$$\boxed{A\vec{x} = x_1\vec{v}_1 + \dots + x_m\vec{v}_m.}$$

That is, $A\vec{x}$ is the <u>linear combination</u> of the <u>columns</u> of A with coefficients given by the entries of $\vec{x}$.

This leads to

<u>Theorem</u> Let A be an $n \times m$ matrix.

THEORY

Then $NS(A) = \vec{0}$

$\Updownarrow$

the equation $A\vec{x} = \vec{0}$ has <u>only</u> $\vec{x} = \vec{0}$ as a solution

$\Updownarrow$

the columns of A are <u>linearly independent</u>

$\Updownarrow$

an REF of A has <u>no free variables</u>

Example: Are the vectors

$$\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}$$ linearly independent?

TECHNIQUE PRACTICE

Solution: **Step 1:** Write the matrix A whose columns are the given vectors

$$A = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & -1\end{pmatrix}$$

Step 2: Find an REF of A

$$\begin{pmatrix}\boxed{1} & 1 & 0\\0 & 1 & 1\\1 & 0 & -1\end{pmatrix} \xrightarrow{R3 \leftarrow R3 - R1} \begin{pmatrix}\boxed{1} & 1 & 0\\0 & \boxed{1} & 1\\0 & -1 & -1\end{pmatrix}$$

$$\big\downarrow R3 \leftarrow R3 + R2$$

$$\begin{pmatrix}\boxed{1} & 1 & 0\\0 & \boxed{1} & 1\\0 & 0 & 0\end{pmatrix}$$

Step 3: This REF has a free variable (x_3). Thus $NS(A)$ has vectors other than $\vec{0}$, so the given vectors are linearly dependent.

Null space and uniqueness of solutions:

$NS(A) = \vec{0} \iff$ for each $\vec{b}$, the equation $A\vec{x} = \vec{b}$ has no more than one solution

Proof ($\Rightarrow$) If $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{b}$, then $A(\vec{x} - \vec{y}) = \vec{0} \Rightarrow \vec{x} - \vec{y}$ in $NS(A) \Rightarrow \vec{x} - \vec{y} = \vec{0}$
$\Rightarrow \vec{x} = \vec{y}$. □

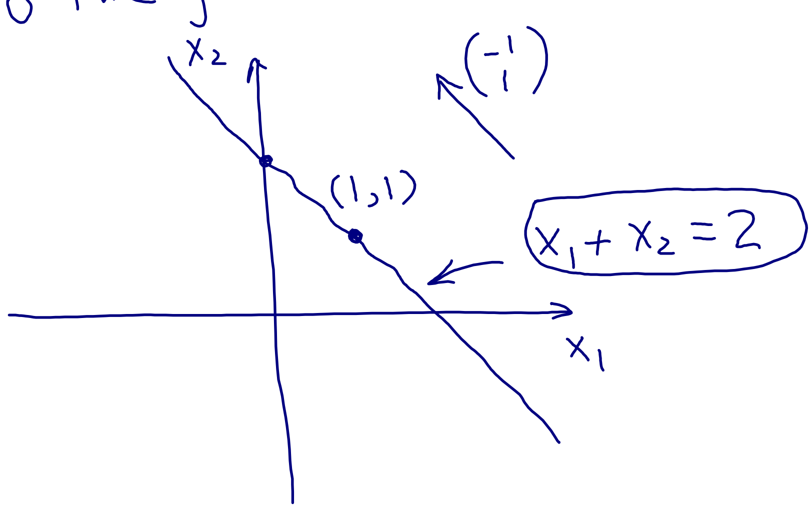
In general, if $A\vec{x} = \vec{b}$ has at least one solution $\vec{x}_0$, then the general solution to $A\vec{x} = \vec{b}$ has the form $\boxed{\vec{x} = \vec{x}_0 + \vec{y}}$ where $\vec{y}$ is an arbitrary element of $NS(A)$

Example: $A = \begin{pmatrix} 1 & 1 \end{pmatrix}$, $\vec{b} = 2$, $A\vec{x} = \vec{b} \iff x_1 + x_2 = 2$

Basis of $NS(A)$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (check it!)

One solution to $A\vec{x} = \vec{b}$ is $\vec{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So the general solution is $\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, c arbitrary



§7.2.2. Column space, rank

Definition Let A be an $n\times m$ matrix.

The column space of A, denoted by $\boxed{CS(A)}$, consists of vectors of the form $A\vec{x}$ where $\vec{x}$ is an arbitrary vector in $\mathbb{R}^m$.

- In other words, a vector $\vec{b}$ lies in $CS(A)$ if and only if the equation $A\vec{x}=\vec{b}$ has a solution

$\boxed{\text{THEORY}}$

- Another interpretation of the column space of A: if $A=(\vec{v}_1 \dots \vec{v}_m)$ then
  $CS(A)=\mathrm{Span}(\vec{v}_1,\dots,\vec{v}_m)$

  That is, $CS(A)$ is the span of the columns of A.

  (To see this, recall that
  $\mathrm{span}(\vec{v}_1,\dots,\vec{v}_m)=$ the set of all linear combinations of $\vec{v}_1,\dots,\vec{v}_m$

  $A\vec{x}=x_1\vec{v}_1+\dots+x_m\vec{v}_m$.)

The column space $CS(A)$ is a subspace of $\mathbb{R}^n$:

$\vec{b}_1=A\vec{x}_1$, $\vec{b}_2=A\vec{x}_2 \Rightarrow \vec{b}_1+\vec{b}_2=A(\vec{x}_1+\vec{x}_2) \Rightarrow \vec{b}_1+\vec{b}_2$ in $CS(A)$

$\vec{b}=A\vec{x}$, c in $\mathbb{R} \Rightarrow c\cdot\vec{b}=A(c\cdot\vec{x}) \Rightarrow c\cdot\vec{b}$ in $CS(A)$

Definition The rank of A is the dimension of the column space of A.

ALGORITHM for finding a basis of CS(A)

TECHNIQUE

Step 1 Use elimination to convert A to row echelon form, which we denote B.

Step 2 Look at the pivot columns of B (those with a pivot element in them). The corresponding columns of A (not of B!) form a basis for CS(A).

Note: rank (A) = number of pivot columns of B

Recall: dim NS(A) = number of free variables in B
= number of non-pivot columns of B

Together these two imply the

Rank-Nullity Theorem:

for any matrix A,

$$\#(\text{columns of } A) = \underbrace{\dim CS(A)}_{\text{this is the rank of } A} + \underbrace{\dim NS(A)}_{\text{this is the nullity of } A}$$

Exercises Find bases for $NS(A)$, $CS(A)$ and check that the rank-nullity theorem holds for the following matrices:

ⓐ $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ ⓑ $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix}$

Solutions: ⓐ Find an REF for A:

$$\begin{pmatrix} \boxed{1} & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{R2 \leftarrow R2 - R1} \begin{array}{cc} x_1 & x_2 \end{array} \begin{pmatrix} \boxed{1} & 1 \\ 0 & 0 \end{pmatrix}$$

x_1 dependent variable (pivot column)
x_2 free variable (nonpivot column)

General solution to $A\vec{x} = 0$:

$$x_1 = -x_2, \quad x_2 \text{ any} \Rightarrow \vec{x} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is a basis for $NS(A)$ $\quad \dim NS(A) = 1$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a basis for $CS(A)$ $\quad \dim CS(A) = 1$

(took the 1st column which is a pivot column)

$$\dim NS(A) + \dim CS(A) = 2 = \# \text{ columns of } A$$

ⓑ Previously found REF for A:

$$\begin{pmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & \boxed{-1} & -2 & -3 \end{pmatrix}$$

x_1, x_2 dependent variables (pivot columns)
x_3, x_4 free variables (nonpivot columns)

Found a basis for $NS(A)$ before: $\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$ $\quad \dim NS(A) = 2$

Basis for $CS(A)$: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\quad \dim CS(A) = 2$

$$\dim NS(A) + \dim CS(A) = 4 = \# \text{columns of } A$$

Column space and existence of solutions:

$CS(A) = \mathbb{R}^n \iff$ for each $\vec{b}$,
the equation $A\vec{x} = \vec{b}$ has <u>at least one solution</u>
$\iff \operatorname{rank}(A) = n$ $\boxed{\text{THEORY}}$

Together with facts about null space we had before we get for an $\boxed{n \times m}$ matrix A

(existence)
$A\vec{x} = \vec{b}$ has ≥ 1 solution for all $\vec{b} \iff \boxed{\operatorname{rank} A = n}$
$A\vec{x} = \vec{b}$ has ≤ 1 solution for all $\vec{b} \iff \boxed{\operatorname{rank} A = m}$
(uniqueness)

In particular to have <u>both</u> existence and uniqueness we need $\operatorname{rank} A = n = m$, so A has to be a <u>square</u> matrix (# rows = # columns)

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§7.2.3. Invertible and singular matrices

Let A be a square matrix of size $n \times n$. Then:

$\operatorname{rank} A = n \iff NS(A) = \vec{0} \iff CS(A) = \mathbb{R}^n$
$\iff$ any REF of A has n pivots

If the above conditions hold, we say A is <u>invertible</u>.
Otherwise we say A is <u>singular</u>.

If A is invertible, then the equation $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ for each $\vec{b}$.

THEORY

This solution has the form $\boxed{\vec{x} = A^{-1}\vec{b}}$

where A^{-1} is the inverse of A, that is, the unique $n \times n$ matrix such that

$$\boxed{A \cdot A^{-1} = I = A^{-1} \cdot A}$$

here I = identity matrix

TECHNIQUE

ALGORITHM for computing A^{-1}:

Step 1: Write the matrix $(A \mid I)$ where I is the identity matrix

Step 2: Apply row operations to bring A to RREF. This RREF has to be the identity matrix (otherwise A is singular)

Now the right half of $(A \mid I)$ is transformed into A^{-1}:

$$(A \mid I) \xrightarrow[\text{Operations}]{\text{Row}} (I \mid A^{-1})$$

Example: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} \boxed{1} & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \xrightarrow{R2 \leftarrow R2 - 3\cdot R1} \left(\begin{array}{cc|cc} \boxed{1} & 2 & 1 & 0 \\ 0 & \boxed{-2} & -3 & 1 \end{array}\right)$$

$$\Big\downarrow R2 \leftarrow -\frac{R2}{2}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right) \xleftarrow{R1 \leftarrow R1 - 2\cdot R2} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right)$$

So $A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$

Check that $A \cdot A^{-1} = A^{-1} \cdot A = I$.

For 2×2 matrices there is a useful formula

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A}^{-1} = \frac{1}{\underbrace{ad-bc}_{\det A}} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Swap diagonal elements
switch sign for
off-diagonal ones

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$