Here we learn how to answer the following questions for a given matrix $A$:

- For which vectors $\vec{b}$ does $A\vec{x} = \vec{b}$ have a solution? (using column space)
- Does $A\vec{x} = \vec{b}$ have a unique solution? (using null space)

And related questions for a set of vectors $\vec{x}_1, \ldots, \vec{x}_m$ in $\mathbb{R}^n$:

- Are these vectors linearly independent? (using null space)
- What is the dimension of $\text{span}(\vec{x}_1, \ldots, \vec{x}_m)$? (using rank)
- Find a basis for $\text{span}(\vec{x}_1, \ldots, \vec{x}_m)$ (using column space)

We also study invertible matrices $A$, for which $A\vec{x} = \vec{b}$ has a unique solution $\vec{x} = A^{-1}\vec{b}$ for every $\vec{b}$

and the rank-nullity theorem

All of these will use Row Echelon Form + Gaussian elimination.
§ 7.2.1. Null space

Recall from § 5.3 that for an \( n \times m \) matrix \( A \), the null space of \( A \) is the set of all solutions \( x \) in \( \mathbb{R}^m \) to the equation

\[
A\vec{x} = \vec{0}
\]

We denote the null space of \( A \) by \( \text{NS}(A) \).

Recall also the definition: A subset \( S \) of \( \mathbb{R}^m \) is called a subspace of \( \mathbb{R}^m \), if:

- For all \( \vec{x}, \vec{y} \) in \( S \), the vector \( \vec{x} + \vec{y} \) is also in \( S \)
  (\( S \) is closed under addition)
- For all \( \vec{x} \) in \( S \), \( c \) in \( \mathbb{R} \), the vector \( c\vec{x} \) is also in \( S \)
  (\( S \) is closed under multiplication by scalars)

Note: \( S \) is a subspace of \( \mathbb{R}^2 \) if:

- \( S = \{ \vec{0} \} \), or
- \( S \) = line through \( \vec{0} \), or
- \( \mathbb{R}^2 \)

For any \( n \times m \) matrix \( A \), the set \( \text{NS}(A) \) is a subspace of \( \mathbb{R}^m \):

\[
\begin{align*}
\{ \vec{0} \} &
\quad \Rightarrow A\vec{0} = \vec{0} \\
\text{NS}(A) &
\quad \Rightarrow A(\vec{x} + \vec{y}) = \vec{0} \\
A\vec{x} = \vec{0} &
\quad \Rightarrow A(c\vec{x}) = \vec{0}
\end{align*}
\]
Algorithm for finding basis & dimension

Technique of the null space of a matrix A:

Step 1: Convert (A | 0) to REF using elimination
  Note: the null space stays the same

Step 2: find the general solution to A\vec{x} = \vec{0}, depending on arbitrary values of the free variables

Step 3: Write the formula for the general solution as a linear combination with free variables as coefficients.
  The vectors in the linear combination form a basis for the null space NS(A)

Note: dimension of NS(A) (number of elements in its basis)
  # of free variables in the REF of A

Example 1: A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}

Step 1: Bring (A | 0) to REF
  \begin{pmatrix} 2 & 1 & 1 & | & 0 \\ 1 & 1 & 0 & | & 0 \end{pmatrix}
  \xrightarrow{R2 \leftarrow R2 - \frac{R1}{2}} \begin{pmatrix} 2 & 1 & 1 & | & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & | & 0 \end{pmatrix}
Step 2: Find the general solution to $A\vec{x} = \vec{0}$: use augmented matrix $(A | \vec{0})$

\[
\begin{bmatrix}
2 & 1 & 1 & | & 0 \\
0 & 1 & -1/2 & | & 0 \\
\end{bmatrix}
\]

$x_1, x_2$ dependent variables

$x_3$ free variable

\[
\begin{align*}
2x_1 + x_2 + x_3 &= 0 \\
\frac{1}{2}x_2 - \frac{1}{2}x_3 &= 0 \\
\end{align*}
\]

Get $x_2 = x_3$, $x_1 = -\frac{x_2 + x_3}{2} = -x_3$

General solution: $\begin{cases} x_1 = -x_3 \\ x_2 = x_3 \\ x_3 \text{ any} \end{cases}$

Step 3: Write $\vec{x} = \begin{pmatrix} -x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

Basis for $\text{NS}(A)$: $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, dimension $(\text{NS}(A)) = 1$.

Geometric picture: the intersection of two planes in $\mathbb{R}^3$ defined by the equations $2x_1 + x_2 + x_3 = 0$ and $x_1 + x_2 = 0$ is the line spanned by the vector $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. 

\[2x_1 + x_2 + x_3 = 0 \quad \quad x_1 + x_2 = 0\]
Example 2: \[ A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 0 \end{pmatrix} \]

Step 1: \( \begin{pmatrix} 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 5 & 0 \end{pmatrix} \) \( \xrightarrow{R2 \leftarrow R2 - 4R1} \) \( \begin{pmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & -1 & -2 & -3 & 0 \end{pmatrix} \)

Let's actually convert this to RREF:

\( \begin{pmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & -1 & -2 & -3 & 0 \end{pmatrix} \) \( \xrightarrow{R2 \leftarrow R2} \) \( \begin{pmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{pmatrix} \)

\( \downarrow \) \( R1 \leftarrow R1 - 2 \cdot R2 \)

\( \begin{pmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{pmatrix} \)

Step 2: \( X_1, X_2 \) dependent variables

\( X_3, X_4 \) free variables

\[
\begin{cases}
X_1 - X_3 - 2X_4 = 0 \\
X_2 + 2X_3 + 3X_4 = 0
\end{cases}
\]

General solution to \( AX = 0 \):

\[
\begin{cases}
X_1 = X_3 + 2X_4 \\
X_2 = -2X_3 - 3X_4
\end{cases}
\]

\( X_3, X_4 \) any

Step 3: Write the general solution above as

\[
\vec{x} = \begin{pmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}
\]

Basis for \( \text{NS}(A) \): \( \begin{pmatrix} 1/2 \\ -2/3 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 0 \end{pmatrix} \), \( \text{dim} \ (\text{NS}(A)) = 2 \).
The null space lets us find out when a given set of vectors is linearly independent, using the following fact: If $A = [\vec{v}_1 \ldots \vec{v}_m]$ is an $n \times m$ matrix with columns $\vec{v}_1, \ldots, \vec{v}_m$, then for any vector $\vec{x} = (x_1 \ldots x_m)$ we have

$$A\vec{x} = x_1\vec{v}_1 + \ldots + x_m\vec{v}_m.$$ 

That is, $A\vec{x}$ is the linear combination of the columns of $A$ with coefficients given by the entries of $\vec{x}$.

This leads to

**Theorem:** Let $A$ be an $n \times m$ matrix.

Then $\text{NS}(A) = \{0\}$

the equation $A\vec{x} = \vec{0}$ has only $\vec{x} = \vec{0}$ as a solution

the columns of $A$ are linearly independent

an REF of $A$ has no free variables
Example: Are the vectors 
\[
\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}
\] linearly independent?

Solution: Step 1: Write the matrix $A$ whose columns are the given vectors 
\[
A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}
\]

Step 2: Find an $\text{REF}$ of $A$
\[
\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\]

Step 3: This $\text{REF}$ has a free variable ($x_3$). Thus $\text{NS}(A)$ has vectors other than $\vec{0}$, so the given vectors are linearly dependent.
Null space and uniqueness of solutions:

\[ \text{NS}(A) = \overrightarrow{0} \iff \text{for each } \overrightarrow{b}, \]
the equation \( A\overrightarrow{x} = \overrightarrow{b} \) has no more than one solution.

\[ \text{Proof} (=) \text{If } A\overrightarrow{x} = \overrightarrow{b} \text{ and } A\overrightarrow{y} = \overrightarrow{b}, \text{ then } \]
\[ A(\overrightarrow{x} - \overrightarrow{y}) = \overrightarrow{0} \implies \overrightarrow{x} - \overrightarrow{y} \in \text{NS}(A) \implies \overrightarrow{x} - \overrightarrow{y} = \overrightarrow{0} \]
\[ \implies \overrightarrow{x} = \overrightarrow{y}. \]

In general, if \( A\overrightarrow{x} = \overrightarrow{b} \) has at least one solution \( \overrightarrow{x}_0 \), then the general solution to \( A\overrightarrow{x} = \overrightarrow{b} \) has the form \( \overrightarrow{x} = \overrightarrow{x}_0 + \overrightarrow{y} \), where \( \overrightarrow{y} \) is an arbitrary element of \( \text{NS}(A) \).

Example: \( A = \begin{pmatrix} 1 & 1 \\ \end{pmatrix}, \overrightarrow{b} = 2 \), \( A\overrightarrow{x} = \overrightarrow{b} \iff x_1 + x_2 = 2 \)

Basis of \( \text{NS}(A) \) is \( \begin{pmatrix} -1 \\ 1 \end{pmatrix} \) (check it!)

One solution to \( A\overrightarrow{x} = \overrightarrow{b} \) is \( \overrightarrow{x}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \)

So the general solution is \( \overrightarrow{x} = \overrightarrow{x}_0 + c \begin{pmatrix} -1 \\ 1 \end{pmatrix} \),

\[ c \text{ arbitrary.} \]
§ 7.2.2. Column space, rank

**Definition.** Let \( A \) be an \( n \times m \) matrix. The **column space** of \( A \), denoted by \( \text{CS}(A) \), consists of vectors of the form \( \overline{X} A \) where \( \overline{X} \) is an arbitrary vector in \( \mathbb{R}^n \).

- In other words, a vector \( \overline{b} \) lies in \( \text{CS}(A) \) if and only if the equation \( A \overline{X} = \overline{b} \) has a solution.

- Another interpretation of the column space of \( A \): if \( A = (\overline{V}_1 \ldots \overline{V}_m) \) then \( \text{CS}(A) = \text{span}(\overline{V}_1, \ldots, \overline{V}_m) \).

That is, \( \text{CS}(A) \) is the span of the columns of \( A \).

(To see this, recall that \( \text{span}(\overline{V}_1, \ldots, \overline{V}_m) = \) the set of all linear combinations of \( \overline{V}_1, \ldots, \overline{V}_m \).

\[ A \overline{X} = x_1 \overline{V}_1 + \ldots + x_m \overline{V}_m. \]

The column space \( \text{CS}(A) \) is a subspace of \( \mathbb{R}^n \):

- \( \overline{b}_1 = A \overline{X}_1, \overline{b}_2 = A \overline{X}_2 \Rightarrow \overline{b}_1 + \overline{b}_2 = A (\overline{X}_1 + \overline{X}_2) \Rightarrow \overline{b}_1 + \overline{b}_2 \) in \( \text{CS}(A) \)

- \( \overline{b} = A \overline{x}, \overline{c} \in \mathbb{R} \Rightarrow c \cdot \overline{b} = A (c \overline{x}) \Rightarrow c \cdot \overline{b} \) in \( \text{CS}(A) \)

**Definition.** The **rank** of \( A \) is the dimension of the column space of \( A \).
Algorithm for finding a basis of $\text{CS}(A)$

**Technique**

Step 1 Use elimination to convert $A$ to row echelon form, which we denote $B$.

Step 2 Look at the pivot columns of $B$ (those with a pivot element in them). The corresponding columns of $A$ (not of $B$!) form a basis for $\text{CS}(A)$.

Note: $\text{rank}(A) =$ number of pivot columns of $B$

Recall: $\dim \text{NS}(A) =$ number of free variables in $B$

$\dim \text{NS}(A) =$ number of non-pivot columns of $B$

Together these two imply the **Rank-Nullity Theorem**:

\[
\#(\text{columns of } A) = \dim \text{CS}(A) + \dim \text{NS}(A)
\]

This is the rank of $A$

This is the nullity of $A$
Exercises: Find bases for $\text{NS}(A)$, $\text{CS}(A)$ and check that the rank-nullity theorem holds for the following matrices:

@ $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$  
@ $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix}$

Solutions:

@ Find an REF for $A$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$x_1$ dependent variable (pivot column)  
$x_2$ free variable (non-pivot column)

General solution to $AX = 0$:

$x_1 = -x_2$, $x_2$ any  
$\Rightarrow \mathbf{x} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$(-1, 1)$ is a basis for $\text{NS}(A)$  
$\dim \text{NS}(A) = 1$

$(1, 1)$ is a basis for $\text{CS}(A)$  
$\dim \text{CS}(A) = 1$

(We took the 1st column which is a pivot column)

$\#	ext{ columns of } A = 2$

@ Previously found REF for $A$:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & -3 \end{pmatrix}$$

$x_1, x_2$ dependent variables (pivot columns)  
$x_3, x_4$ free variables (non-pivot columns)

Found a basis for $\text{NS}(A)$ before:

$$\begin{pmatrix} -2 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$\dim \text{NS}(A) = 2$

Basis for $\text{CS}(A)$: $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}$  
$\dim \text{CS}(A) = 2$

$\#	ext{ columns of } A = 4$
Column space and existence of solutions:

\[ \text{CS}(A) = \mathbb{R}^n \iff \text{for each } \vec{b}, \]
the equation \( A\vec{x} = \vec{b} \) has \underline{at least one solution} \( \iff \text{rank } (A) = n \)

Together with facts about null space we had before we get for an \( n \times m \) matrix \( A \)

\[ A\vec{x} = \vec{b} \] has \( \geq 1 \) solution for all \( \vec{b} \) \( \iff \text{rank } A = n \)
\[ A\vec{x} = \vec{b} \] has \( \leq 1 \) solution for all \( \vec{b} \) \( \iff \text{rank } A = m \)

(existence)

(uniqueness)

In particular to have both existence and uniqueness we need \( \text{rank } A = n = m \), so \( A \) has to be a \underline{square} matrix (\# rows = \# columns)

§7.2.3. Invertible and singular matrices

Let \( A \) be a square matrix of size \( n \times n \). Then:

\[ \text{rank } A = n \iff \text{NS}(A) = \{0\} \iff \text{CS}(A) = \mathbb{R}^n \]
\[ \iff \text{any REF of } A \text{ has } n \text{ pivots } \]

If the above conditions hold, we say \( A \) is \underline{invertible}.

Otherwise we say \( A \) is \underline{singular}.
If $A$ is invertible, then the equation $A\overrightarrow{x} = \overrightarrow{b}$ has a unique solution $\overrightarrow{x}$ for each $\overrightarrow{b}$. This solution has the form 

$$\overrightarrow{x} = A^{-1}\overrightarrow{b}$$

where $A^{-1}$ is the inverse of $A$, that is, the unique nxn matrix such that

$$A \cdot A^{-1} = I = A^{-1} \cdot A$$

**Algorithm for computing $A^{-1}$:**

**Step 1:** Write the matrix $(A \mid I)$ where $I$ is the identity matrix.

**Step 2:** Apply row operations to bring $A$ to RREF. This RREF has to be the identity matrix (otherwise $A$ is singular).

Now the right half of $(A \mid I)$ is transformed into $A^{-1}$:

$$(A \mid I) \xrightarrow{\text{Row Operations}} (I \mid A^{-1})$$
Example: \( A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \)

\[
\begin{pmatrix}
\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}
& 1 \\
0 & 1
\end{pmatrix}
\xrightarrow{R_2 \leftarrow R_2 - 3 \cdot R_1}
\begin{pmatrix}
\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}
& 1 \\
0 & -2
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
0 & 3/2
\end{pmatrix}
\xrightarrow{R_1 \leftarrow R_1 - 2 \cdot R_2}
\begin{pmatrix}
\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}
& 1 \\
0 & 1
\end{pmatrix}
\]

So \( A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} \)

Check that \( A \cdot A^{-1} = A^{-1} \cdot A = I \).

For 2x2 matrices there is a useful formula:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\]

Swap diagonal elements

Switch sign for off-diagonal ones

Example:

\[
\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}
\]