§7. LINEAR ALGEBRA II

§7.1. Systems of linear equations and elimination

We study here linear systems of the form $\boxed{A\vec{x} = \vec{b}}$ where

- A is a given $n \times m$ matrix
- $\vec{b}$ is a given vector of length n
- $\vec{x}$ is an unknown vector of length m

Vector equations of the form above are equivalent to linear systems of n equations in m variables.

Exercise Convert the equation $A\vec{x} = \vec{b}$ with $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$ to a system of linear equations.

PRACTICE TECHNIQUE

Solution Write $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Now write the variables x_1, x_2, x_3 over columns of A and convert rows into equations:

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$$

$\downarrow$

$$\begin{array}{c|c|c} x_1 & x_2 & x_3 \\ 1 & 2 & 3 \\ \hline 3 & 4 & 5 \end{array} \quad = \quad \begin{array}{c} \\ 6 \\ 12 \end{array}$$

$\downarrow$

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ 3x_1 + 4x_2 + 5x_3 = 12 \end{cases}$$

Exercise Write the system of equations

$$\begin{cases} x_2 - 2x_1 = 1 \\ 4x_1 - 2x_2 = 2 \end{cases}$$

in vector form

PRACTICE
TECHNIQUE

Solution We arrange the system so that the 1st column is x_1, 2nd is x_2.

Then we find $A, \vec{b}$:

$$\begin{cases} -2 \cdot x_1 + 1 \cdot x_2 = 1 \\ 4 \cdot x_1 - 2 \cdot x_2 = 2 \end{cases} \iff A\vec{x} = \vec{b} \text{ where } \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$A = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

§7.1.1 Augmented matrix & row operations

For the vector equation $A\vec{x} = \vec{b}$ we use the augmented matrix $(A \mid \vec{b})$ (i.e. we put $\vec{b}$ to the right of A and separated it by a vertical line)

E.g. for $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix}\vec{x} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$ the augmented matrix is $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 3 & 4 & 5 & 12 \end{array}\right)$

THEORY
TECHNIQUE

We now introduce row operations.

These are operations on the augmented matrix which do not change the set of solutions $\vec{x}$.

There are 3 kinds of row operations:

① Multiply a row by a nonzero number

Example $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 3 & 4 & 5 & 12 \end{array}\right) \xrightarrow{1^{st} \text{ row } \times 3} \left(\begin{array}{ccc|c} 3 & 6 & 9 & 18 \\ 3 & 4 & 5 & 12 \end{array}\right)$

Note: $\begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ 3x_1 + 4x_2 + 5x_3 = 12 \end{cases} \iff \begin{cases} 3x_1 + 6x_2 + 9x_3 = 18 \\ 3x_1 + 4x_2 + 5x_3 = 12 \end{cases}$

Since the only change is that we multiplied the 1^{st} equation by 3

2 Swap two rows

Example
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 3 & 4 & 5 & 12 \end{array}\right) \xrightarrow{\text{row 1} \leftrightarrow \text{row 2}} \left(\begin{array}{ccc|c} 3 & 4 & 5 & 12 \\ 1 & 2 & 3 & 6 \end{array}\right)$$

This does not change the set of solutions because we merely changed the order of the equations.

3 Add (or subtract) to a row a multiple of another row

Example
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 3 & 4 & 5 & 12 \end{array}\right) \xrightarrow{\text{row 2} \leftarrow \text{row 2} - 3 \cdot \text{row 1}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -2 & -4 & -6 \end{array}\right)$$

Indeed: $3 - 3 \cdot 1 = 0$, $4 - 3 \cdot 2 = -2$,
$5 - 3 \cdot 3 = -4$, $12 - 3 \cdot 6 = -6$

This does not change the set of solutions because assuming that $x_1 + 2x_2 + 3x_3 = 6$ the equations $3x_1 + 4x_2 + 5x_3 = 12$ and $-2x_2 - 4x_3 = -6$ are equivalent.

We can apply row operations one after another:

Example
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 3 & 4 & 5 & 12 \end{array}\right) \xrightarrow{R2 \leftarrow R2 - 3 \cdot R1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -2 & -4 & -6 \end{array}\right)$$
$$\downarrow R1 \leftarrow R1 + R2$$
$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -2 & -4 & -6 \end{array}\right)$$

and we keep the same set of solutions

§7.1.2. Row Echelon Form (REF)

THEORY

Definition. We say that a matrix A is in row echelon form (REF) if:

1. All zero rows of A (if any) are at the bottom of A
2. The first nonzero entry in each nonzero row, which we call the $\boxed{\text{pivot}}$ of this row, is to the right of the pivots of higher rows
3. All entries in a column below a pivot entry are 0.

Examples: $\square$ denotes pivot elements

$\begin{pmatrix} \boxed{1} & 2 & 3 \\ \boxed{3} & 4 & 5 \end{pmatrix}$ NOT REF: $\boxed{3}$ not to the right of $\boxed{1}$

$\begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{-2} & -4 \end{pmatrix}$ REF $\begin{pmatrix} \square & * & * \\ 0 & \square & * \end{pmatrix}$

$\begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{4} \end{pmatrix}$ REF $\begin{pmatrix} \square & * & * \\ 0 & 0 & \square \end{pmatrix}$

We say an augmented matrix $(A \mid \vec{b})$ is in REF if A is in REF.

The reason REF is important is because

1. There is an algorithm to convert any matrix into REF (we present it in §7.1.3 below)
2. It is relatively easy to solve a system $A\vec{x} = \vec{b}$ with A in REF using back substitution

Back substitution: an algorithm to find the general solution to $A\vec{x} = \vec{b}$ with A in REF

Step 1: If $(A \mid \vec{b})$ contains a row $(0 \dots 0 \mid *)$ where $* \neq 0$ then stop: the equation $A\vec{x} = \vec{b}$ has no solutions.

TECHNIQUE

(Indeed, $(0 \dots 0 \mid *)$ corresponds to the equation $0 \cdot x_1 + \dots + 0 \cdot x_m = *$ which cannot be satisfied for nonzero $*$)

Step 2: Identify the pivot columns of A, these correspond to dependent variables. The other variables (columns with no pivot) are called free variables.

Examples:

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{-2} & -4 \end{pmatrix}$$

x_1, x_2 dependent
x_3 free

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{4} \end{pmatrix}$$

x_1, x_3 dependent
x_2 free

Step 3: The free variables can be chosen arbitrarily. The dependent variables are determined uniquely for any choice of free variables by solving the equations in reverse order & using back substitution!

(Note: if there are no free variables then the equation has a unique solution. Otherwise there are infinitely many solutions.)

Example: $\left(\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 6 \\ 0 & \boxed{-2} & -4 & -6 \end{array}\right) \to \begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ \quad -2x_2 - 4x_3 = -6 \end{cases}$

x_3 free variable

2^{nd} equation $\Rightarrow$ $x_2 = 3 - 2x_3$

(substitute x_2)

1^{st} equation $\Rightarrow$ $x_1 = 6 - 2x_2 - 3x_3 =$
$= 6 - 2(3 - 2x_3) - 3x_3$
$= x_3$

So the general solution to the system is
$$\begin{cases} x_1 = x_3 \\ x_2 = 3 - 2x_3 \\ x_3 \text{ any} \end{cases}$$

Note: Since $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 3 & 4 & 5 & 12 \end{array}\right) \xrightarrow[\text{operations}]{\text{row}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -2 & -4 & -6 \end{array}\right)$

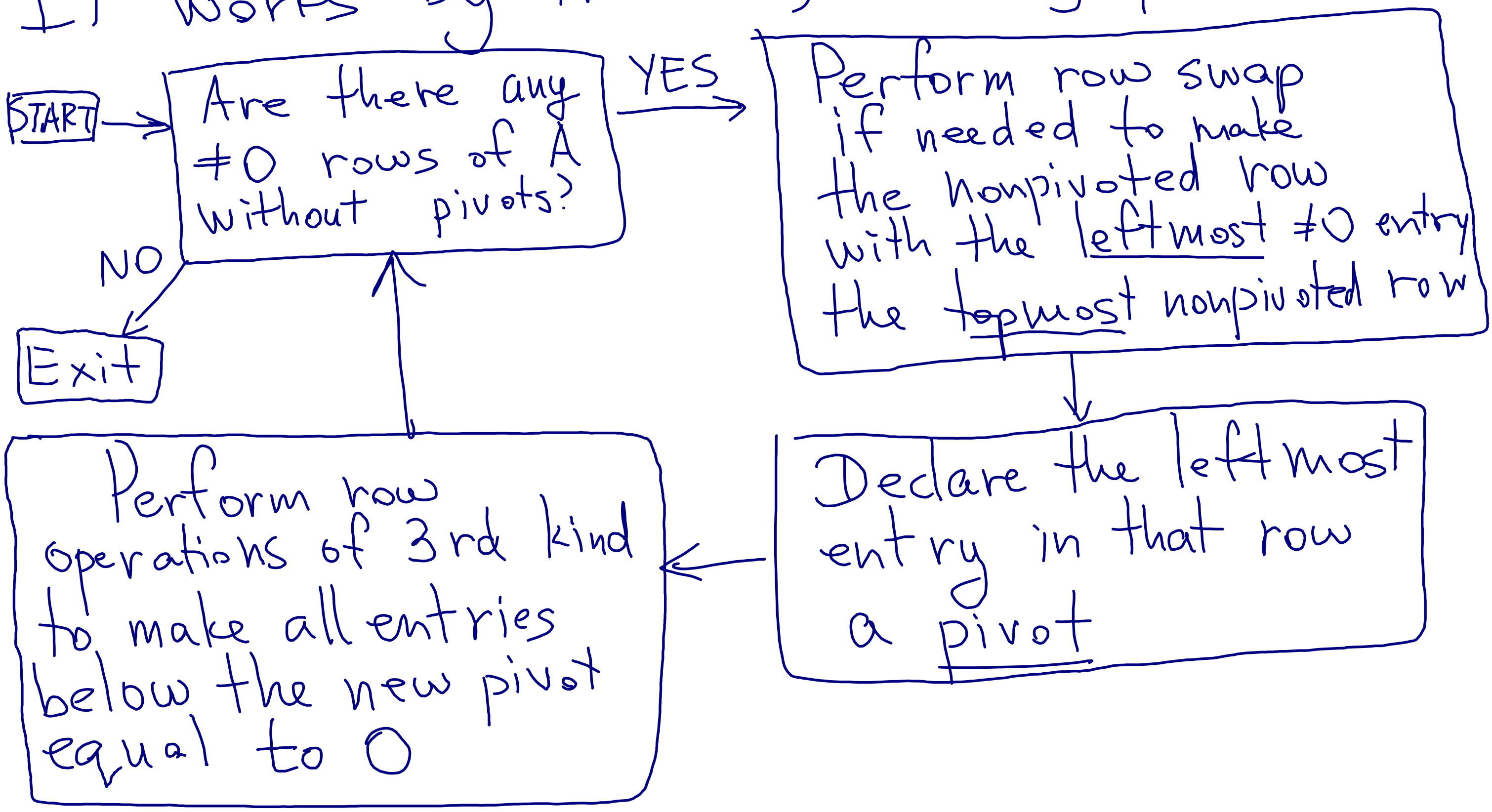
the above also gives the general solution to
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ 3x_1 + 4x_2 + 5x_3 = 12 \end{cases}$$

§7.1.3. Gaussian elimination

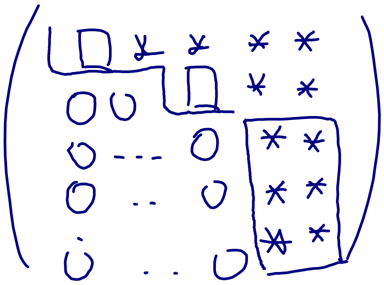
TECHNIQUE

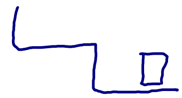
This is an algorithm to convert any augmented system to one in REF.

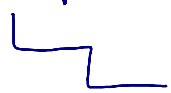
It works by iteration, creating pivots:

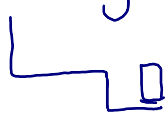


At each step of the algorithm the matrix A looks something like

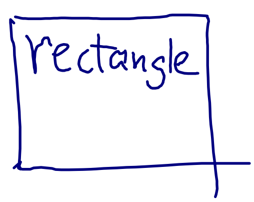


Here □ denotes the pivots already selected, the broken line  ends at the last pivot,

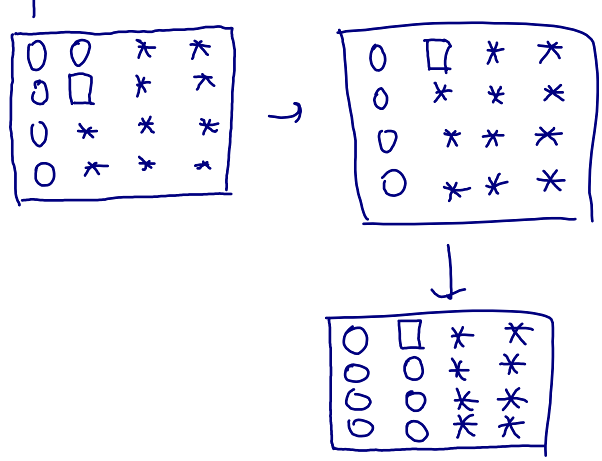
and <u>everything under</u>  <u>consists of zeroes</u>.

To do the next step, we look at the rectangle right & below the last pivot: 

move the leftmost ≠0 element of this rectangle to the top,



call this the next pivot, and clear out the space under it by row operations:



Examples:

1 $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$\downarrow R3 \leftarrow R3 - R1$

$\begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{2} \end{pmatrix} \xleftarrow{R3 \leftarrow R3 + R2} \begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & -1 & 1 \end{pmatrix}$

2 $\left(\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 3 & 6 & 7 & 3 \end{array}\right) \xrightarrow[R3 \leftarrow R3 - 3 \cdot R1]{R2 \leftarrow R2 - 2 \cdot R1} \left(\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right)$

$\downarrow R2 \leftrightarrow R3$

$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1 \\ 0 & 0 & \boxed{-2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Got the system

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ \qquad\qquad -2x_3 = 0 \end{cases}$$

x_2 free variable,

$x_3 = 0$, $x_1 = 1 - 2x_2$

Get $\begin{cases} x_1 = 1 - 2x_2 \\ x_2 \text{ any} \\ x_3 = 0 \end{cases}$

3 $\left(\begin{array}{cc|c} \boxed{2} & 2 & 7 \\ 4 & 3 & 8 \\ 6 & 1 & 7 \end{array}\right) \xrightarrow[R3 \Leftarrow R3 - 3\cdot R1]{R2 \Leftarrow R2 - 2\cdot R1} \left(\begin{array}{cc|c} \boxed{2} & 2 & 7 \\ 0 & \boxed{-1} & -6 \\ 0 & -5 & -14 \end{array}\right)$

$\downarrow R3 \Leftarrow R3 - 5\cdot R2$

$\left(\begin{array}{cc|c} \boxed{2} & 2 & 7 \\ 0 & \boxed{-1} & -6 \\ 0 & 0 & 16 \end{array}\right)$

No solution
since got the line
$(0 \quad 0 \mid 16)$
which corresponds to $0\cdot x_1 + 0\cdot x_2 = 16$.

SEE $\boxed{\text{MIT}_x\ 3.1.11}$ FOR A LONGER EXAMPLE

§7.1.4. Reduced Row Echelon Form (RREF)

A matrix A is in RREF if:

- A is in REF
- Each pivot value $= 1$
- In each pivot column all elements are $= 0$ except the pivot element.

TECHNIQUE

By performing row operations we can convert REF to RREF:

- divide rows by pivot elements to make pivot values $= 1$
- do row operations of 3^{rd} kind to wipe out each pivot column

Example:

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 6 \\ 0 & \boxed{-2} & -4 & -6 \end{array}\right) \xrightarrow{R2 \to -\frac{R2}{2}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \end{array}\right)$$

$$\Big\downarrow R1 \leftarrow R1 - 2\cdot R2$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & 2 & 3 \end{array}\right)$$

The <u>advantage</u> of RREF over REF is that the solution is <u>easier to find</u>.

Basically, <u>RREF</u> = <u>REF</u> + <u>back substitution</u>

E.g. for the above example

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \end{array}\right) \text{ gives } \begin{cases} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 3 \end{cases}$$

Which immediately leads to

$$\begin{cases} x_1 = x_3 \\ x_2 = 3 - 2x_3 \\ x_3 \text{ any} \end{cases}$$