§6.4. Stability

Definition We say that a system of ODEs $\vec{y}' = A\vec{y}$ is:

THEORY

- **stable**, if every solution $\vec{y}$ satisfies $\vec{y}(t) \to \vec{0}$ as $t \to \infty$
- **semistable**, if every solution $\vec{y}$ stays bounded as $t \to \infty$ but some solutions $\vec{y}$ do not converge to $\vec{0}$ as $t \to \infty$;
- **unstable**, if some solutions $\vec{y}$ are not bounded (in fact they will $\to \infty$) as $t \to \infty$.

Recall that when A is diagonalizable, the general (complex) solution is

$$\vec{y}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2$$

where λ_1, λ_2 are the eigenvalues of A

We have $|e^{\lambda_1 t}| = e^{(\mathrm{Re}\,\lambda_1)\cdot t}$

As $t \to \infty$, we have $|e^{\lambda_1 t}| \to \begin{cases} \infty, & \text{if } \mathrm{Re}\,\lambda_1 > 0 \\ 1, & \text{if } \mathrm{Re}\,\lambda_1 = 0 \\ 0, & \text{if } \mathrm{Re}\,\lambda_1 < 0 \end{cases}$

For stability we need both $|e^{\lambda_1 t}|, |e^{\lambda_2 t}| \to 0$.

For semistability we need
$|e^{\lambda_1 t}|, |e^{\lambda_2 t}| \not\to \infty$.

If A is not diagonalizable then the general solution features $e^{\lambda t}$, $te^{\lambda t}$ where λ is the eigenvalue of A.

This gives

$\boxed{\text{TECHNIQUE}}$

Theorem (Stability in terms of eigenvalues)

Consider the system $\vec{y}\,' = A\vec{y}$ and let λ_1, λ_2 be the eigenvalues of A. Then:

- If $\operatorname{Re}\lambda_1, \operatorname{Re}\lambda_2$ are both <0 then the system is $\boxed{\text{stable}}$
- If at least one of $\operatorname{Re}\lambda_1, \operatorname{Re}\lambda_2$ is >0 then the system is $\boxed{\text{unstable}}$
- If one of $\operatorname{Re}\lambda_1, \operatorname{Re}\lambda_2$ is $=0$ and the other is ≤ 0 then:
  - A is diagonalizable $\Rightarrow$ the system is $\boxed{\text{Semistable}}$
  - A is not diagonalizable $\Rightarrow$ the system is $\boxed{\text{unstable}}$

The last case only happens if A has double eigenvalue 0 but $A \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, e.g. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

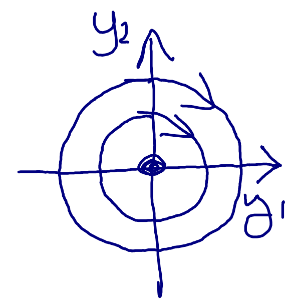
Examples: (using companion system)

PRACTICE

1 Harmonic oscillator:

$y'' + y = 0$, $\lambda_1 = i$, $\lambda_2 = -i$, $\mathrm{Re}\,\lambda_1 = \mathrm{Re}\,\lambda_2 = 0$

$\boxed{\text{Semistable}}$.



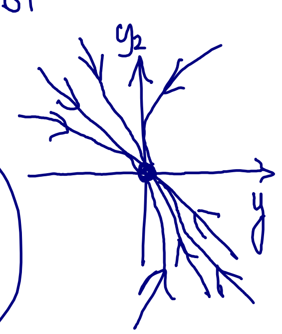
Indeed, $\vec{y}(t) = \begin{pmatrix} C_1 \cos t + C_2 \sin t \\ -C_1 \sin t + C_2 \cos t \end{pmatrix}$

Solutions are bounded but they do not go to 0 (unless $C_1 = C_2 = 0$)

2 Damped harmonic oscillator:

$y'' + 3y' + 2y = 0$, $\lambda_1 = -1$, $\lambda_2 = -2$ both < 0

$\boxed{\text{Stable}}$



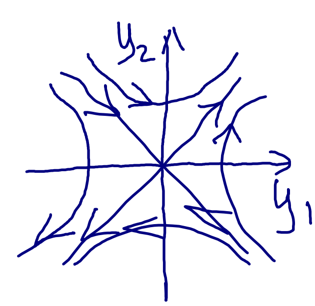
Indeed, $\vec{y}(t) = \begin{pmatrix} C_1 e^{-t} + C_2 e^{-2t} \\ -C_1 e^{-t} - 2C_2 e^{-2t} \end{pmatrix}$

All solutions $\to 0$ as $t \to \infty$

3 $y'' - y = 0$, $\lambda_1 = -1$, $\lambda_2 = 1$.

One of the eigenvalues is > 0

$\boxed{\text{Unstable}}$



Indeed, $\vec{y}(t) = \begin{pmatrix} C_1 e^{t} + C_2 e^{-t} \\ C_1 e^{t} - C_2 e^{-t} \end{pmatrix}$

goes to ∞ if $C_1 \neq 0$

As in §4.6.3 we can characterize stability in terms of the coefficients of the characteristic polynomial, which is

$$P(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + (\det A):$$

<u>Theorem (Stability in terms of tr-det)</u>

The system $\vec{y}' = A\vec{y}$ is <u>stable</u> if and only if

$$\boxed{\operatorname{tr} A < 0, \ \det A > 0}$$

The system $\vec{y}' = Ay$ is <u>semistable</u>, if and only if

either $\operatorname{tr} A < 0$, $\det A = 0$

or $\operatorname{tr} A = 0$, $\det A > 0$

or $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

same as saying $\operatorname{tr} A = 0$, $\det A = 0$, A diagonalizable.

<u>Proof</u> Using Vieta's formulas similarly to §4.6.3. See the picture on the next page. □

On the tr-det plane: (ev means eigenvalue)

det

$\boxed{\text{Semistable}}$ if tr $=0$, det >0:
e.v. $\pm iq$

$\boxed{\text{Stable}}$ if
tr $A<0$, det $A>0$:
either real e.v. <0
or complex e.v.
with real part <0

$\boxed{\text{Unstable}}$
if tr $A>0$
since $\lambda_1+\lambda_2=\text{tr} A$
at least one of
$\text{Re}\,\lambda_1, \text{Re}\,\lambda_2$ is >0

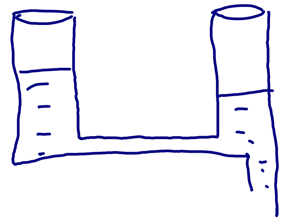
tr

$\boxed{\text{Semistable}}$ if tr <0,
det $=0$:
one ev $=0$, another <0

if tr $=$ det $=0$:
double e.v. $\lambda=0$
$\boxed{\text{Semistable}}$ if $A=\begin{pmatrix}0&0\\0&0\end{pmatrix}$
$\boxed{\text{Unstable}}$ otherwise

$\boxed{\text{Unstable}}$ if det <0
(one e.v. <0, one e.v. >0)

Example: in §6.1.3 we studied connected tanks



where

$$A = \begin{pmatrix} -a & a \\ a & -a-b \end{pmatrix}$$

and $a > 0$ (connecting pipe)
$b \geq 0$ (leakage)

For which a, b is it (semi)stable?

Solution: $\operatorname{tr} A = -2a - b < 0$
$\det A = ab \geq 0$

Stable when $b > 0$

Semistable when $b = 0$

Here is a plot of the matrix $A = \begin{pmatrix} -a & a \\ a & -a-b \end{pmatrix}$ on the tr-det plane when $a = 1$:

$\operatorname{tr} = -2 - b$
$\det = b$