

§6.4. Stability

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Definition We say that a system of ODEs

THEORY

$$\vec{y}' = A\vec{y} \text{ is:}$$

- stable, if every solution \vec{y} satisfies $\vec{y}(t) \rightarrow \vec{0}$ as $t \rightarrow \infty$
- semistable, if every solution \vec{y} stays bounded as $t \rightarrow \infty$ but some solutions \vec{y} do not converge to $\vec{0}$ as $t \rightarrow \infty$;
- unstable, if some solutions \vec{y} are not bounded (in fact they will $\rightarrow \infty$) as $t \rightarrow \infty$.

Recall that when A is diagonalizable,

the general (complex) solution is

$$\vec{y}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

where λ_1, λ_2 are the eigenvalues of A

We have $|e^{\lambda_1 t}| = e^{(\operatorname{Re} \lambda_1) \cdot t}$

As $t \rightarrow \infty$, we have $|e^{\lambda_1 t}| \rightarrow \begin{cases} \infty, & \text{if } \operatorname{Re} \lambda_1 > 0 \\ 1, & \text{if } \operatorname{Re} \lambda_1 = 0 \\ 0, & \text{if } \operatorname{Re} \lambda_1 < 0 \end{cases}$

For stability we need both $|e^{\lambda_1 t}|, |e^{\lambda_2 t}| \rightarrow 0$.

For semistability we need

$$|e^{\lambda_1 t}|, |e^{\lambda_2 t}| \not\rightarrow \infty$$

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If A is not diagonalizable then the general solution features $e^{\lambda t}$, $te^{\lambda t}$ where λ is the eigenvalue of A .

This gives

TECHNIQUE

Theorem (Stability in terms of eigenvalues)

Consider the system $\vec{y}' = A\vec{y}$ and let λ_1, λ_2 be the eigenvalues of A . Then:

- If $\operatorname{Re} \lambda_1, \operatorname{Re} \lambda_2$ are both < 0 then the system is stable
- If at least one of $\operatorname{Re} \lambda_1, \operatorname{Re} \lambda_2$ is > 0 then the system is unstable
- If one of $\operatorname{Re} \lambda_1, \operatorname{Re} \lambda_2$ is $= 0$ and the other is ≤ 0 then:
 - * A is diagonalizable \Rightarrow the system is semistable
 - * A is not diagonalizable \Rightarrow the system is unstable

Examples: (using companion system)

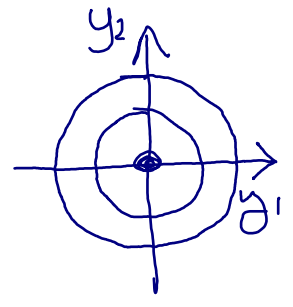
PRACTICE

① Harmonic oscillator:

$y'' + y = 0$, $\lambda_1 = i, \lambda_2 = -i$, $\text{Re } \lambda_1 = \text{Re } \lambda_2 = 0$

Semistable

Indeed, $\vec{y}(t) = \begin{pmatrix} C_1 \cos t + C_2 \sin t \\ -C_1 \sin t + C_2 \cos t \end{pmatrix}$



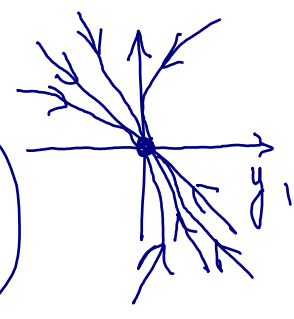
Solutions are bounded but they do not go to 0 (unless $C_1 = C_2 = 0$)

② Damped harmonic oscillator:

$y'' + 3y' + 2y = 0$, $\lambda_1 = -1, \lambda_2 = -2$ both < 0

Stable

Indeed, $\vec{y}(t) = \begin{pmatrix} C_1 e^{-t} + C_2 e^{-2t} \\ -C_1 e^{-t} - 2C_2 e^{-2t} \end{pmatrix}$



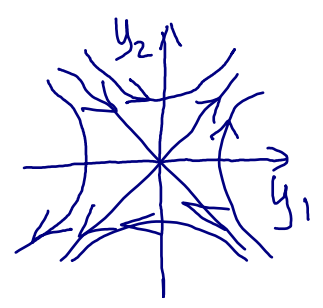
All solutions $\rightarrow 0$ as $t \rightarrow \infty$

③ $y'' - y = 0$, $\lambda_1 = -1, \lambda_2 = 1$.

One of the eigenvalues is > 0

Unstable

Indeed, $\vec{y}(t) = \begin{pmatrix} C_1 e^t + C_2 e^{-t} \\ C_1 e^t - C_2 e^{-t} \end{pmatrix}$



goes to ∞ if $C_1 \neq 0$

As in §4.6.3 we can characterize stability in terms of the coefficients of the characteristic polynomial, which is

$$P(\lambda) = \lambda^2 - (\text{tr} A)\lambda + (\det A):$$

Theorem (Stability in terms of tr-det)

The system $\vec{y}' = A\vec{y}$ is stable if and only if

$$\text{tr} A < 0, \det A > 0$$

The system $\vec{y}' = A\vec{y}$ is semistable, if and only if

either
or
or

$$\begin{matrix} \text{tr} A < 0, \det A = 0 \\ \text{tr} A = 0, \det A < 0 \\ A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{matrix}$$

same as saying
 $\text{tr} A = 0, \det A = 0,$
 A diagonalizable.

Proof Using Vieta's formulas

Similarly to §4.6.3. See the picture on the next page. \square

On the tr-det plane: (ev means eigenvalue)

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det ↑

Semistable

if $\text{tr} = 0, \det > 0$:
e.v. $\pm iq$

$\det = \frac{(\text{tr})^2}{4}$

double e.v.

Stable

if

$\text{tr} A < 0, \det A > 0$:
either real e.v. < 0
or complex e.v.
with real part < 0

Unstable

if $\text{tr} A > 0$
since $\lambda_1 + \lambda_2 = \text{tr} A$
at least one of
 $\text{Re} \lambda_1, \text{Re} \lambda_2$ is > 0

Semistable

if $\text{tr} < 0,$
 $\det = 0$:
one e.v. = 0, another < 0

if $\text{tr} = \det = 0$:
double e.v. $\lambda = 0$

Semistable if $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

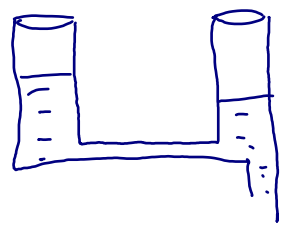
Unstable otherwise

Unstable

if $\det < 0$
(one e.v. $< 0,$ one e.v. > 0)

tr →

Example: in §6.1.3 we studied connected tanks



where

$$A = \begin{pmatrix} -a & a \\ a & -a-b \end{pmatrix} \quad \text{and } a > 0 \text{ (connecting i.e.)}$$

$$b \geq 0 \text{ (leakage)}$$

For which a, b is it (semi)stable?

Solution: $\text{tr } A = -2a - b < 0$
 $\det A = ab \geq 0$

Stable when $b > 0$

Semistable when $b = 0$

Here is a plot of the matrix $A = \begin{pmatrix} -a & a \\ a & -a-b \end{pmatrix}$ on the $\text{tr}-\det$ plane when $a = 1$: $\text{tr} = 2 - b$
 $= b$