§6.3. Phase portraits

§6.3.1. Plotting trajectories

We are (still) studying 2×2 systems

$$\boxed{\vec{y}\,'(t) = A\vec{y}(t)} \qquad (*)$$

We will learn how to plot trajectories of this system

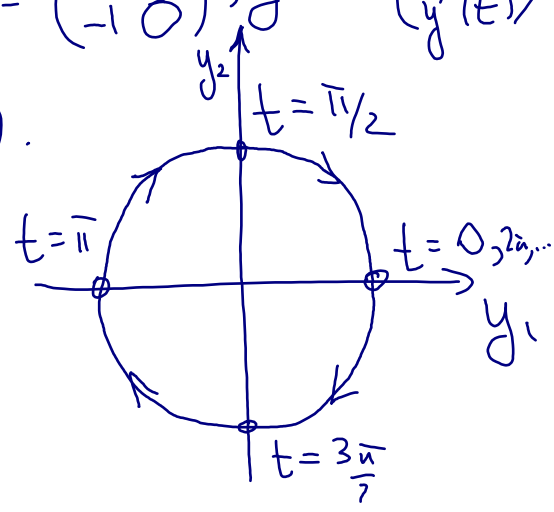
THEORY

Definition: Let $\vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ be a solution to $(*)$. The trajectory of $\vec{y}$ is the curve tracing the point with coordinates $(y_1(t), y_2(t))$ as t varies in $\mathbb{R}$. We call the (y_1, y_2) plane the phase plane.

Example: harmonic oscillator $y'' + y = 0$

Companion system $\vec{y}\,' = A\vec{y}$, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\vec{y}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$

One solution is $\vec{y}(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$.



The trajectory is the unit circle: We will use arrows to mark the direction of increasing t.

Fact: through each point (A_1, A_2) on the plane passes exactly one trajectory of the system $(*)$.
This trajectory corresponds to the unique solution $\vec{y}$ to $(*)$ which satisfies the initial condition $\vec{y}(0) = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$.

Example: for the system $\vec{y}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{y}$
the trajectory passing through $\boxed{(1,0)}$ corresponds to the solution to

$$\begin{cases} y_1' = y_2 \\ y_2' = -y_1 \\ y_1(0) = 1 \\ y_2(0) = 0 \end{cases} \longrightarrow \boxed{\vec{y}(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}}$$

The trajectory passing through $\boxed{(0,1)}$ corresponds to the solution to

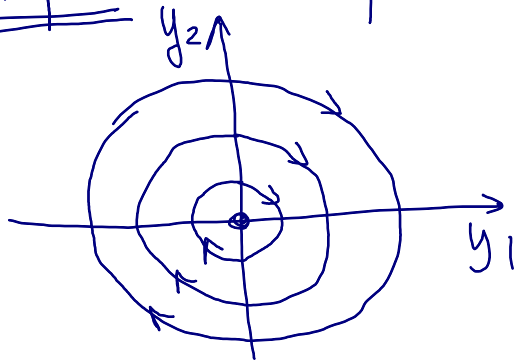
$$\begin{cases} y_1' = y_2 \\ y_2' = -y_1 \\ y_1(0) = 0 \\ y_2(0) = 1 \end{cases} \longrightarrow \boxed{\vec{y}(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}}$$

Both solutions give the same trajectory, the unit circle. They only differ by shifting t by $\pi/2$.

Definition The phase portrait of the system (*) is the plot of all of its trajectories.

In practice we plot only a few trajectories, enough to see what the rest of them look like.

Example: The phase portrait of $\vec{y} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{y}$ is



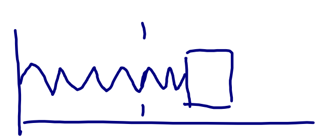
The trajectories are concentric circles $\vec{y} = r\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$, $r > 0$ and the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The origin corresponds to the constant solution $\vec{y}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which is there for all linear systems.

Physical interpretation of the portrait above:

Use the spring model for the harmonic oscillator

$y'' + y = 0$

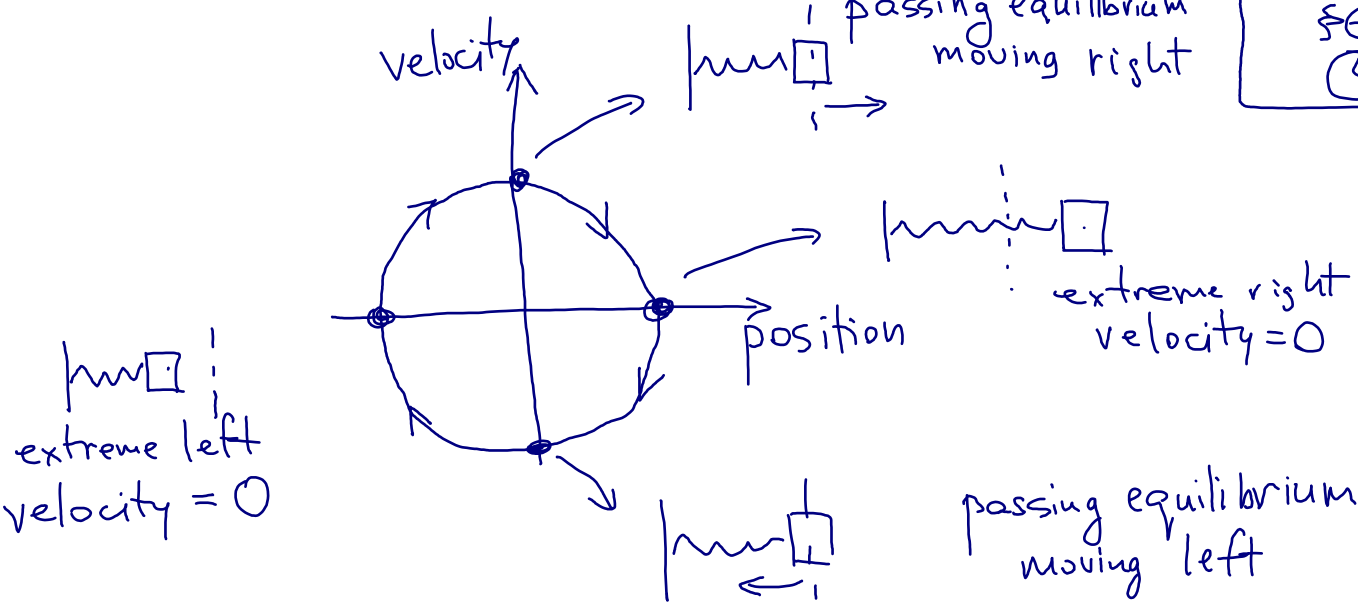


equilibrium point

$y_1 = y$ = position of the spring

$y_2 = y'$ = velocity of the spring

Trajectories are closed $\Rightarrow$ solutions are periodic



§6.3.2. The trace-determinant plane

Our goal now is to learn how to draw phase portraits of $\vec{y}' = A\vec{y}$ for any 2×2 matrix A.

THEORY

The behavior of these depends on the eigenvalues of A, which are the roots of

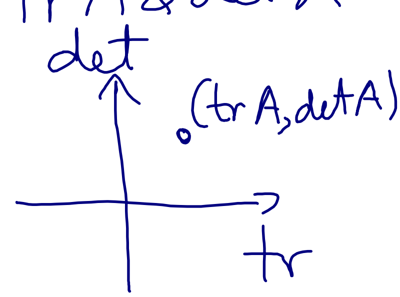
$$P(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A$$

Here $\operatorname{tr} A = a+d$ (trace) if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det A = ad - bc$ (determinant)

So the phase portrait depends on $\operatorname{tr} A$ & $\det A$

We can thus plot A on the trace-determinant plane:



The <u>discriminant</u> of $P(\lambda)$ is

$$D = (\operatorname{tr} A)^2 - 4 \det A$$

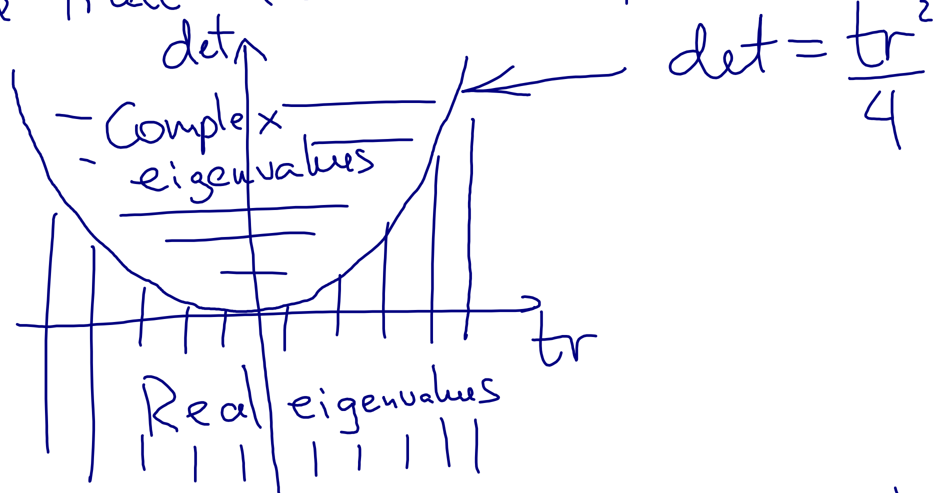
<u>We have:</u>

$\underline{D>0} \iff 2$ real eigenvalues $\iff \underline{(\operatorname{tr} A)^2 > 4 \det A}$

$\underline{D=0} \Rightarrow$ double eigenvalue $\iff \underline{(\operatorname{tr} A)^2 = 4 \det A}$

$\underline{D<0} \Rightarrow 2$ complex eigenvalues $\iff \underline{(\operatorname{tr} A)^2 < 4 \det A}$

We will draw the <u>parabola</u> $\det A = \dfrac{(\operatorname{tr} A)^2}{4}$ on the trace-determinant plane:



If we study a <u>companion system</u> of a 2^{nd} order ODE $y'' + a_1 y' + a_0 y = 0$

then $A = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}$. Thus $\boxed{\begin{aligned} \operatorname{tr} A &= -a_1 \\ \det A &= a_0 \end{aligned}}$

Examples

1. $4y'' + y = 0$ (harmonic oscillator)

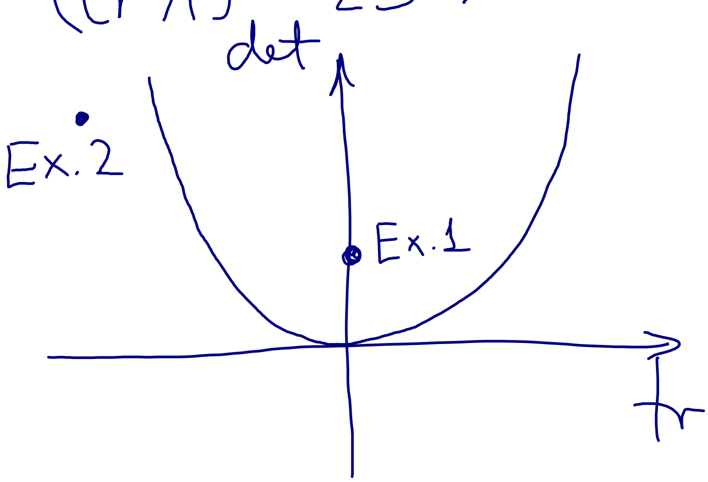
Divide by 4 to get $y'' + \frac{y}{4} = 0$

$a_1 = 0$, $a_0 = \frac{1}{4}$. $\operatorname{tr} A = 0$, $\det A = \frac{1}{4}$.

2. $y'' + 5y + 6 = 0$ (damped harmonic oscillator)

$a_1 = 5$, $a_0 = 6$, $\operatorname{tr} A = -5$, $\det A = 6$.

Note: $(\operatorname{tr} A)^2 = 25 > 24 = 4 \det A$



In the rest of this section we learn how to draw phase portraits case by case & identify these cases on the trace-determinant plane.

§6.3.3. Case 1: distinct real eigenvalues

We consider first the case when A has two real eigenvalues $\boxed{\lambda_1 < \lambda_2.}$

Let $\vec{v}_1, \vec{v}_2$ be the corresponding eigenvectors: $\boxed{A\vec{v}_1 = \lambda_1 \vec{v}_1,\ A\vec{v}_2 = \lambda_2 \vec{v}_2}$

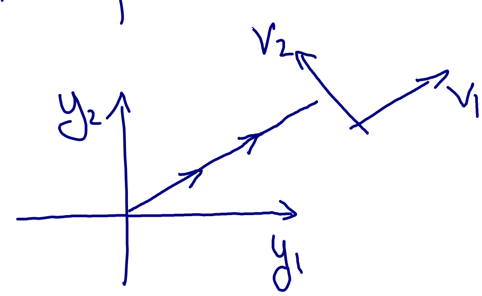
Recall the general solution $\boxed{\vec{y} = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2}$

We consider sub-cases depending on the signs of λ_1, λ_2.

Case 1.1: $\boxed{0 < \lambda_1 < \lambda_2}$

We first draw the trajectories of the special solution

$\vec{y}(t) = e^{\lambda_1 t}\vec{v}_1 \quad (C_1 = 1,\ C_2 = 0)$



This is a ray along $\vec{v}_1$:

Limits: $\vec{y}(t) \longrightarrow \infty$ as $t \to \infty$

$\vec{y}(t) \longrightarrow 0$ as $t \to -\infty$

So the ray is directed outward.

It converges to the origin as $t \to -\infty$ but does not reach it in any finite time.

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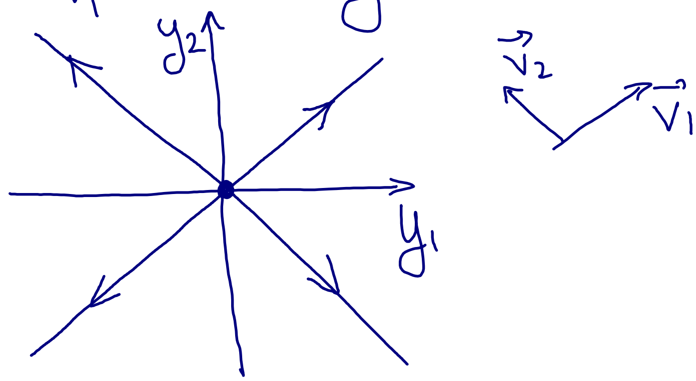
We similarly plot

$\vec{y} = -e^{\lambda_1 t}\vec{v}_1$ and $\vec{y} = \pm e^{\lambda_2 t}\vec{v}_2$,

giving



What about other trajectories?

Take $\vec{y}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2$

where for example $C_1, C_2 > 0$.

Then since $\lambda_1, \lambda_2 > 0$, we still have

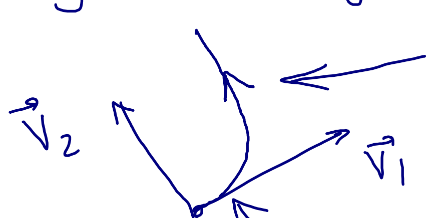
$$\vec{y}(t) \to \begin{cases} \infty, & t \to \infty \\ 0, & t \to -\infty \end{cases}$$

But since $\lambda_1 < \lambda_2$,

as $t \to \infty$ the term $C_2 e^{\lambda_2 t}\vec{v}_2$ dominates

as $t \to -\infty$ the term $C_1 e^{\lambda_1 t}\vec{v}_1$ dominates

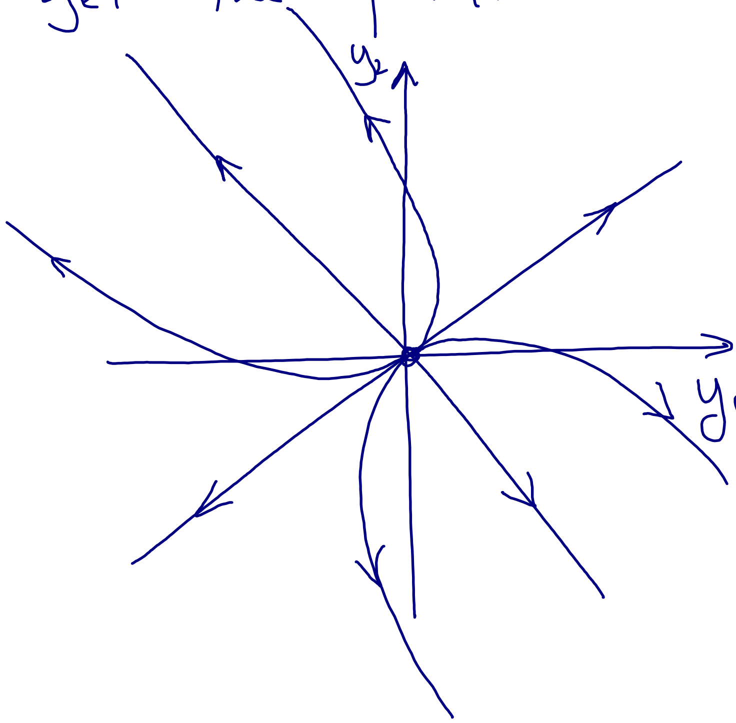
We get a trajectory which looks like

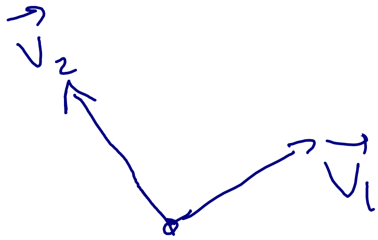


going off to ∞ as $t \to \infty$ along the $\vec{v}_2$ direction

going to 0 as $t \to -\infty$ tangent to $\vec{v}_1$

Putting in the trajectories with other signs of C_1, C_2 we get the portrait





!! The course website will have nicer pictures

This is called a $\boxed{\text{Source node}}$

Example: $\begin{cases} y_1' = y_1 \\ y_2' = 2y_2 \end{cases}$ $\quad A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$\lambda_1 = 1, \ \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 2, \ \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

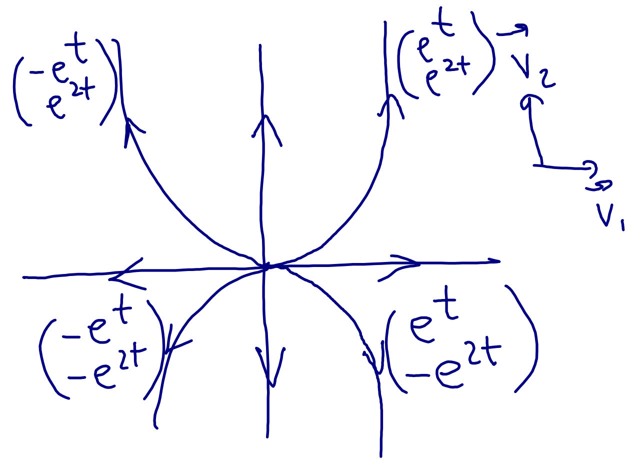
Solution $\vec{y} = \begin{pmatrix} C_1 e^t \\ C_2 e^{2t} \end{pmatrix}$.

E.g. for $C_1 = C_2 = 1$

get $\vec{y} = \begin{pmatrix} e^t \\ e^{2t} \end{pmatrix}$

i.e. $y_1 = e^t, \ y_2 = e^{2t}$

$y_2 = y_1^2$, got a parabola



Case 1.2: $\boxed{\lambda_1 < \lambda_2 < 0}$

This looks very similar to Case 1.1 but with the direction of t reversed.

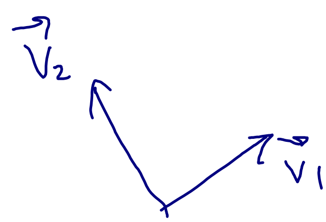
So $\vec{y}(t) \to \begin{cases} 0, & t \to \infty \\ \infty, & t \to -\infty \end{cases}$

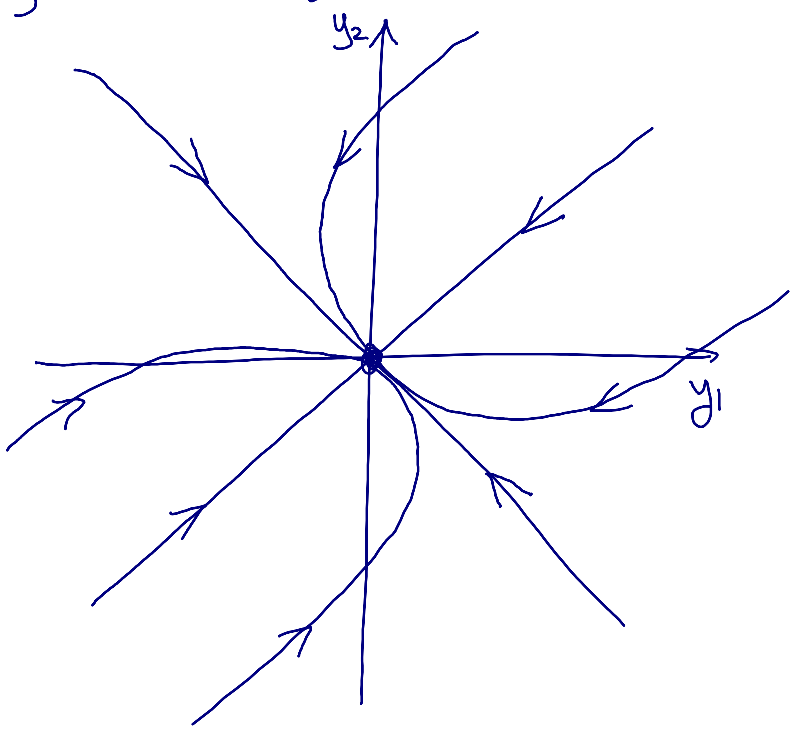
We have $\vec{y}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2$ and since $\lambda_1 < \lambda_2$, if $C_1, C_2 \neq 0$ then

$C_1 e^{\lambda_1 t}\vec{v}_1$ dominates as $t \to -\infty$

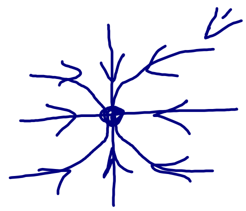
$C_2 e^{\lambda_2 t}\vec{v}_2$ dominates as $t \to \infty$

We get a $\boxed{\text{Sink node}}$





Example: $\begin{cases} y_1' = -2y_1 \\ y_2' = -y_2 \end{cases}$

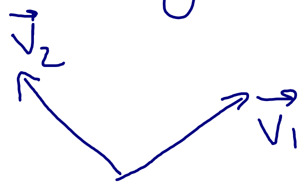


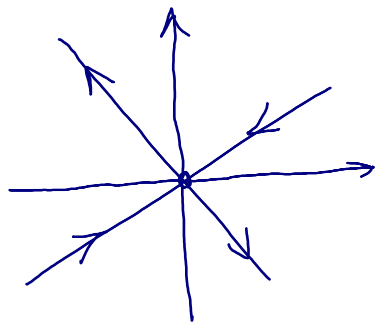
This trajectory

$y_1 = e^{-2t}$
$y_2 = e^{-t}$
$y_1 = y_2^2$

Case 1.3: $\boxed{\lambda_1 < 0 < \lambda_2}$

The special trajectories $\vec{y} = \pm e^{\lambda_1 t}\vec{v}_1$, $\vec{y} = \pm e^{\lambda_2 t}\vec{v}_2$ now look like this:





Indeed,

$$e^{\lambda_1 t}\vec{v}_1 \to \begin{cases} 0, & t \to \infty \\ \infty, & t \to -\infty \end{cases}$$

$$e^{\lambda_2 t}\vec{v}_2 \to \begin{cases} \infty, & t \to \infty \\ 0, & t \to -\infty \end{cases}$$

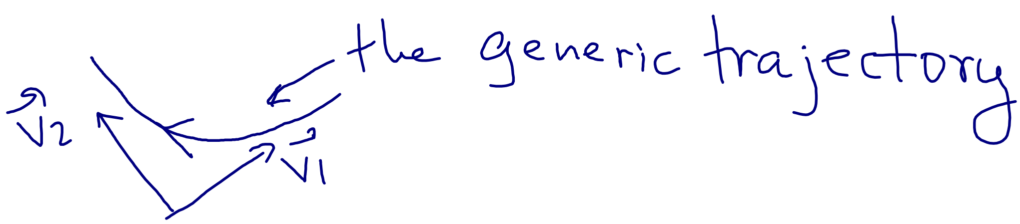
What happens for other trajectories?

$\vec{y} = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2$, $C_1, C_2 \neq 0$

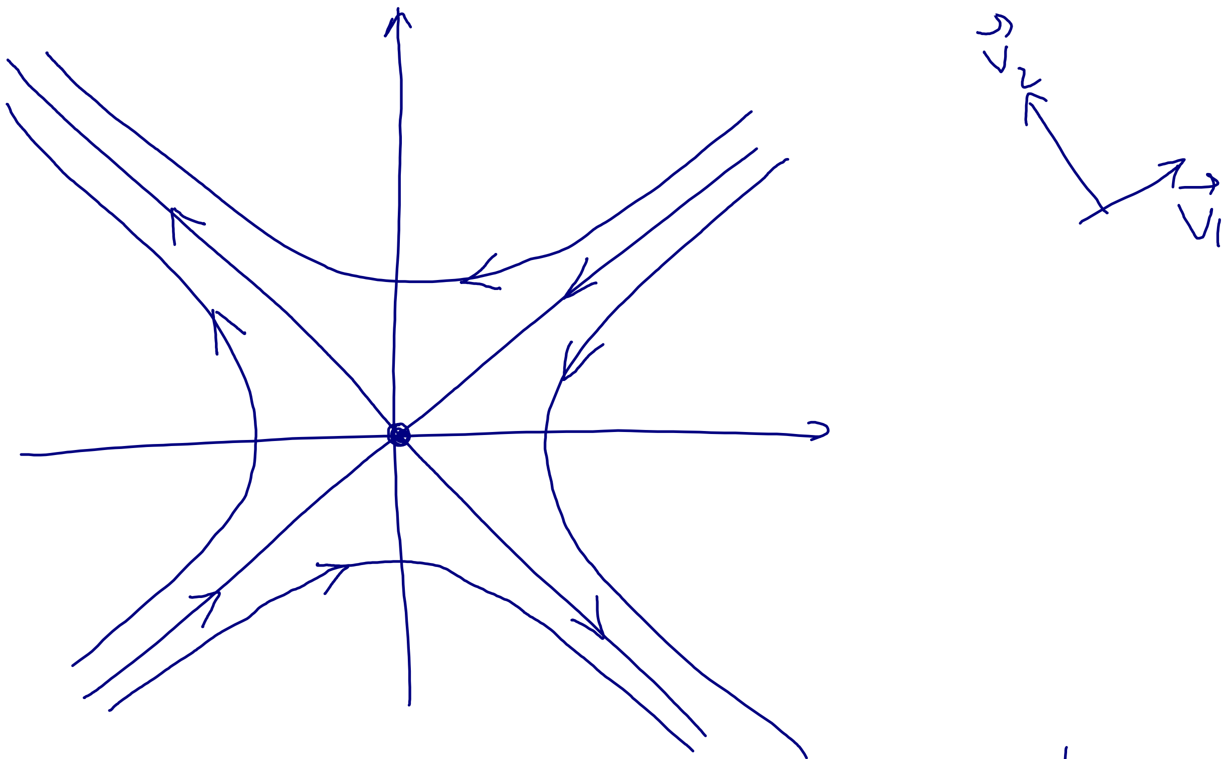
satisfies $\vec{y}(t) \to \infty$ for $t \to \infty$ and for $t \to -\infty$.

As $t \to \infty$, the term $C_1 e^{\lambda_1 t}\vec{v}_1$ goes to 0
the term $C_2 e^{\lambda_2 t}\vec{v}_2$ dominates;
the trajectory is asymptotic to the line through $\vec{v}_2$

As $t \to -\infty$, the term $C_2 e^{\lambda_2 t}\vec{v}_2$ goes to 0
the term $C_1 e^{\lambda_1 t}\vec{v}_1$ dominates,
the trajectory is asymptotic to the line through $\vec{v}_1$:

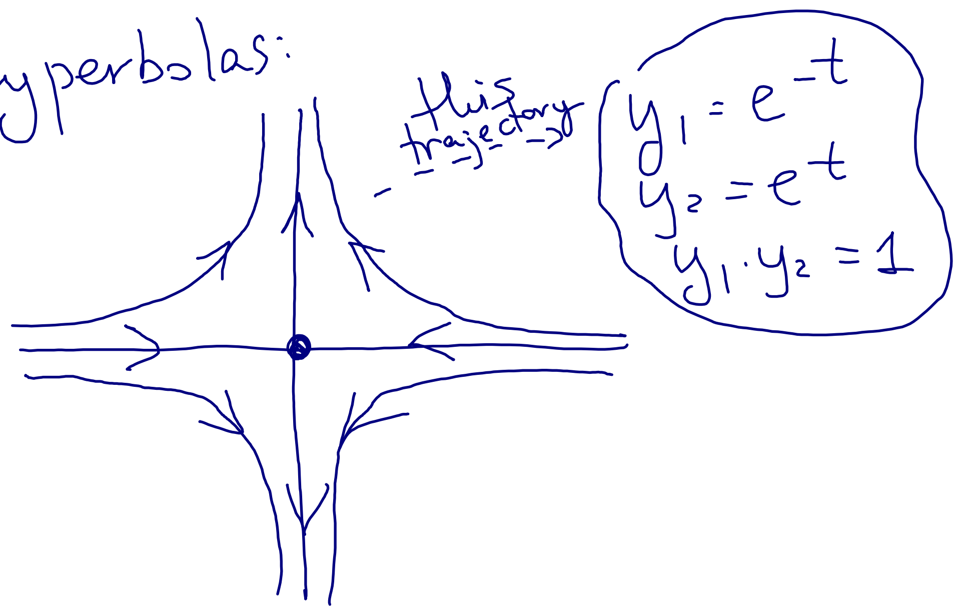


We get a picture known as a $\boxed{\text{saddle}}$:



Example 1: $\begin{cases} y_1' = -y_1 \\ y_2' = y_2 \end{cases} \Rightarrow \begin{matrix} y_1 = C_1 e^{-t} \\ y_2 = C_2 e^{t} \end{matrix}$

Note that in this example the product $y_1 \cdot y_2$ is constant, so most trajectories are hyperbolas:



Example 2:

$$\begin{cases} y_1' = y_2 \\ y_2' = y_1 \end{cases}$$

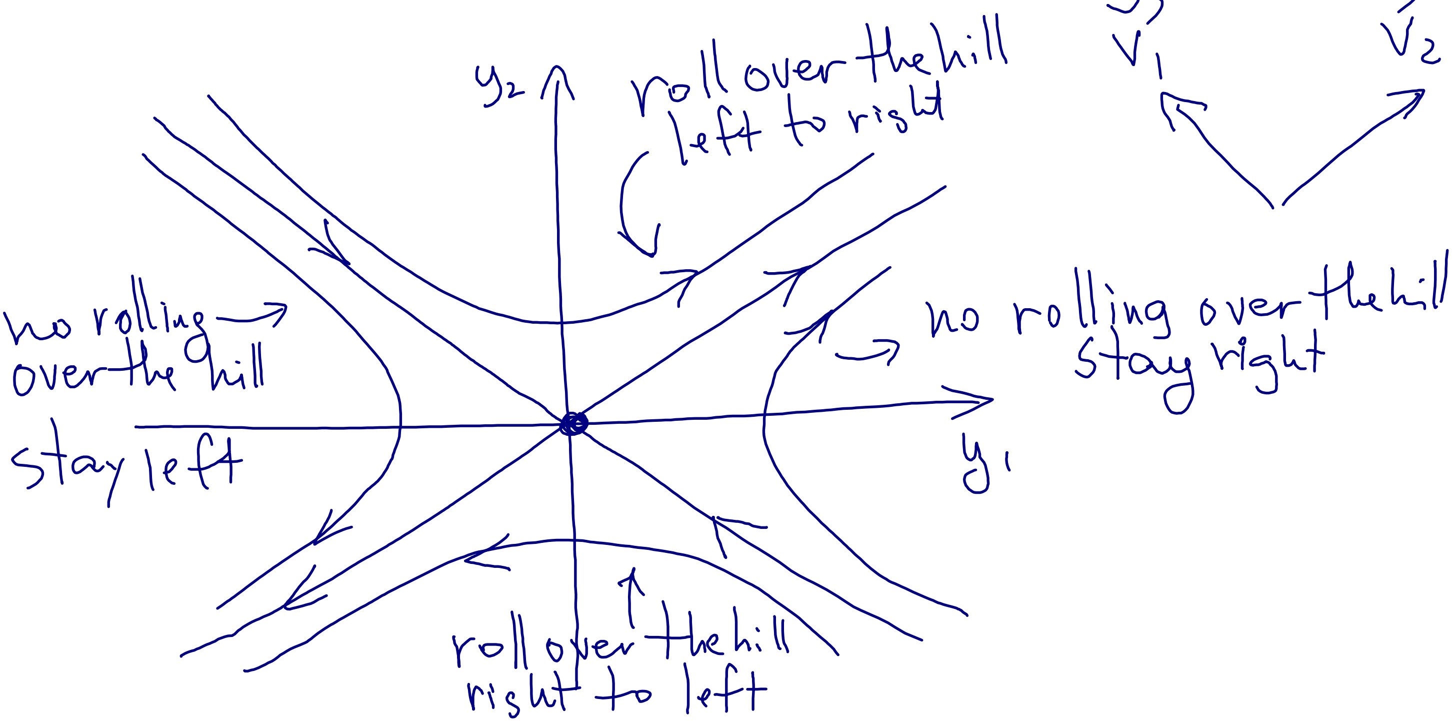
We have $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A\vec{v}_1 = \lambda_1 \vec{v}_1$, $A\vec{v}_2 = \lambda_2 \vec{v}_2$

where $\lambda_1 = -1$, $\lambda_2 = 1$, $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

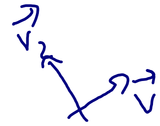
This is the companion system to $y'' - y = 0$.

Physical interpretation: this is a basic model of a point particle near the top of a hill

$y_1 =$ position, $y_2 =$ velocity

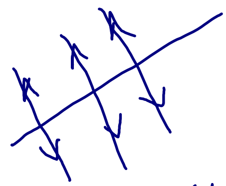


Case 1.4: $\lambda_1 = 0 < \lambda_2$

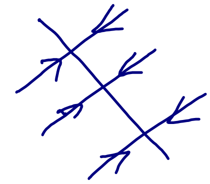


Case 1.5: $\lambda_1 < \lambda_2 = 0$

These give combs



Case 1.4



Case 1.5

but we won't cover these here.
See MITx 2.4.8 for more details

Where do cases 1.1 - 1.3
lie on the trace-determinant plane?
Recall λ_1, λ_2 solve $\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$
By Vieta's formulas

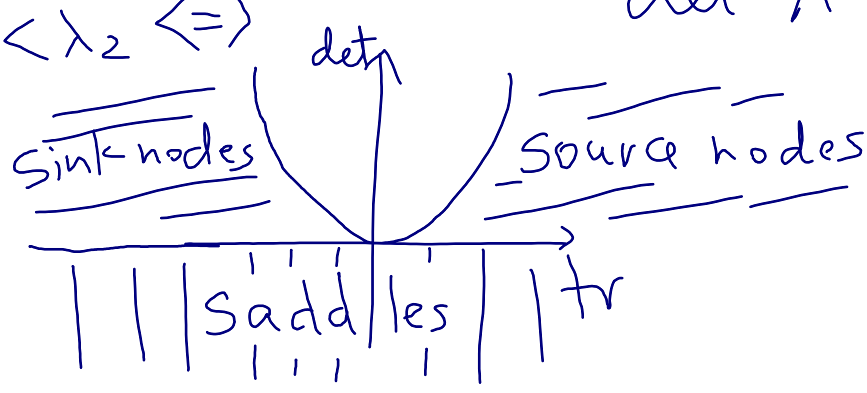
$$\begin{cases} \lambda_1 + \lambda_2 = \operatorname{tr} A \\ \lambda_1 \cdot \lambda_2 = \det A \end{cases}$$

So (assuming $\lambda_1 \neq \lambda_2$ real)

$0 < \lambda_1 < \lambda_2 \iff \operatorname{tr} A > 0, \det A > 0$

$\lambda_1 < \lambda_2 < 0 \iff \operatorname{tr} A < 0, \det A > 0$

$\lambda_1 < 0 < \lambda_2 \iff \det A < 0$



§6.3.4. Case 3: Complex eigenvalues

Now assume that A has complex eigenvalues $\lambda_1 = p + iq$, $\lambda_2 = p - iq$ where $q > 0$.

The general solution to $\vec{y}' = A\vec{y}$ is

$$\boxed{\vec{y}(t) = C_1 \operatorname{Re}(e^{\lambda_1 t}\vec{v}_1) + C_2 \operatorname{Im}(e^{\lambda_1 t}\vec{v}_1)}$$

The vector $\vec{v}_1$ is complex & we write $\boxed{\vec{v}_1 = \vec{a} + i\vec{b}}$ where $\vec{a}, \vec{b}$ are real vectors.

(The vectors $\vec{a}, \vec{b}$ will always form a basis of $\mathbb{R}^2$.)

Case 3.1: $\boxed{p=0}$, thus $\lambda_1 = iq$, $\lambda_2 = -iq$

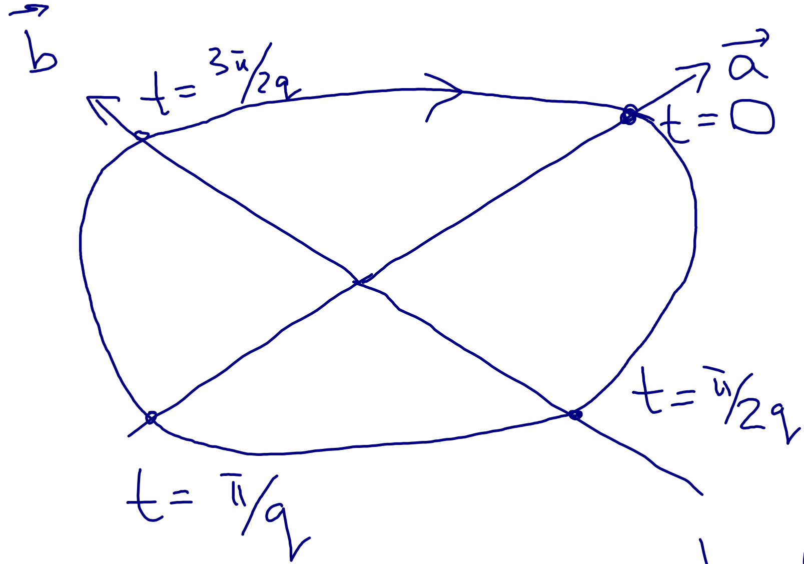
Using Euler's formula we write the solution with $C_1 = 1, C_2 = 0$ as

$$\vec{y}(t) = \operatorname{Re}(e^{iqt}(\vec{a} + i\vec{b}))$$

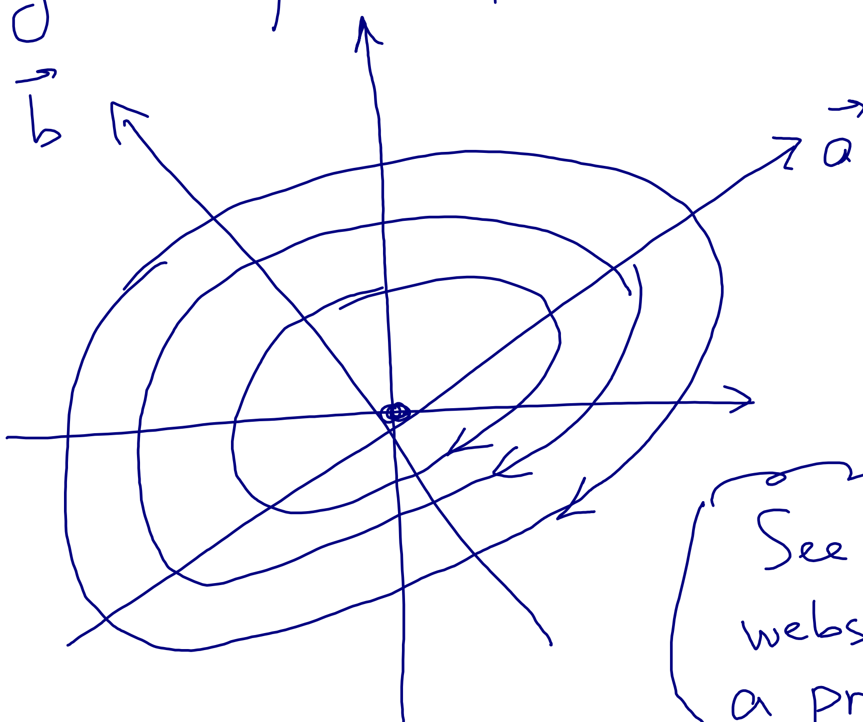
$$= \operatorname{Re}((\cos(qt) + i\sin(qt))(\vec{a} + i\vec{b}))$$

$$= \cos(qt)\cdot\vec{a} - \sin(qt)\cdot\vec{b}.$$

The trajectory is an ellipse:



Other trajectories look similar giving the phase portrait called a $\boxed{\text{center}}$.



See the course website for a prettier picture

Example: harmonic oscillator

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\lambda_1 = i$, $q = 1$, $\vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} = \vec{a} + i\vec{b}$

$\vec{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

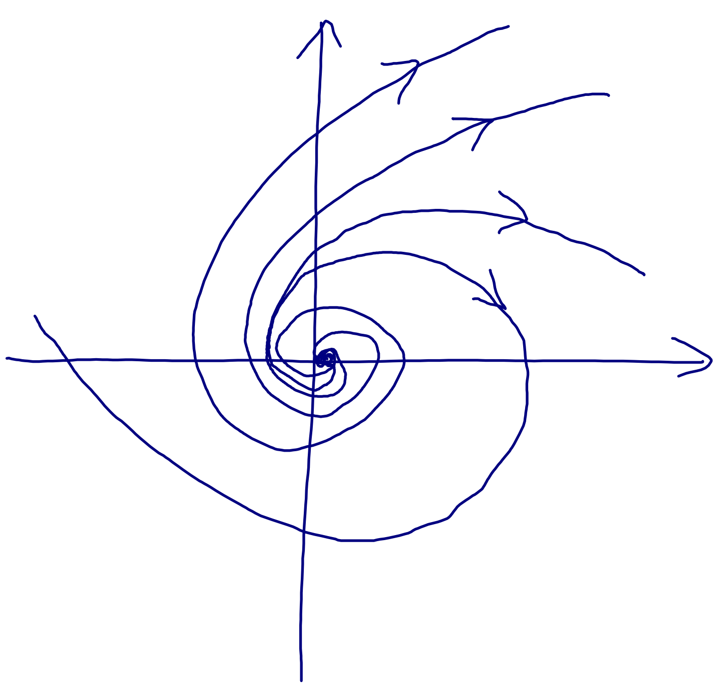
Case 3.2: $\boxed{p>0}$ Recall $\lambda_1 = p+iq$

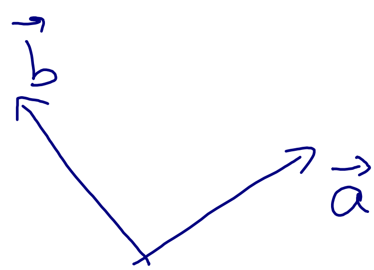
We again look at the specific trajectory

$$\vec{y}(t) = \mathrm{Re}\left(e^{\lambda_1 t}\vec{v}_1\right) = \mathrm{Re}\left(e^{pt}e^{iqt}(\vec{a}+i\vec{b})\right)$$
$$= e^{pt}\left(\cos(qt)\cdot\vec{a} - \sin(qt)\cdot\vec{b}\right).$$

This is similar to the formula for the center except for the prefactor e^{pt}. This prefactor $\longrightarrow \begin{cases}\infty, & t\to\infty\\ 0, & t\to-\infty\end{cases}$

and the trajectory is a spiral.
Other trajectories are similar & this gives the spiral source:

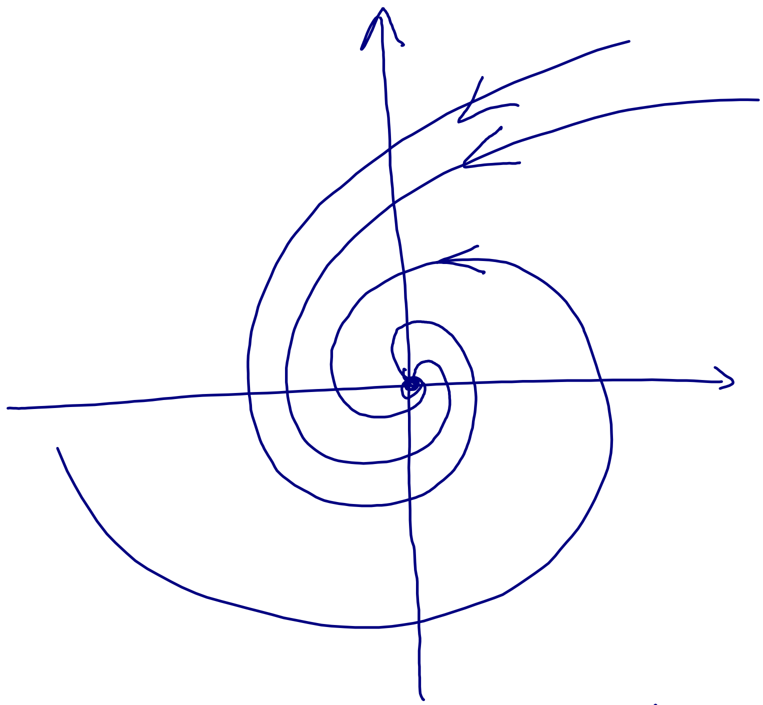




Case 3.3: $\boxed{p<0}$

Now the prefactor $e^{pt} \to \begin{cases} 0, & t\to\infty \\ \infty, & t\to -\infty \end{cases}$

Get the **spiral sink**:

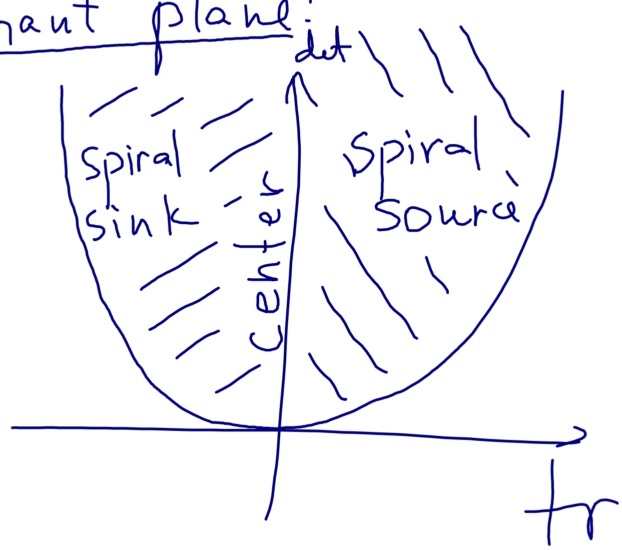


Example: any underdamped harmonic oscillator gives this case.

On the trace-determinant plane:

$\operatorname{tr} A = \lambda_1 + \lambda_2 = 2p$

So $\operatorname{sign}(p) = \operatorname{sign}(\operatorname{tr} A)$



§6.3.5. Case 2: double eigenvalue

We finally consider the case when A has a double (real) eigenvalue
$\boxed{\lambda_1 = \lambda_2 = \lambda.}$

We will only look at the situations when $\boxed{\lambda \neq 0}$ & $\boxed{A \text{ is diagonalizable}}$
(for other cases, see MITx 2.4.9)

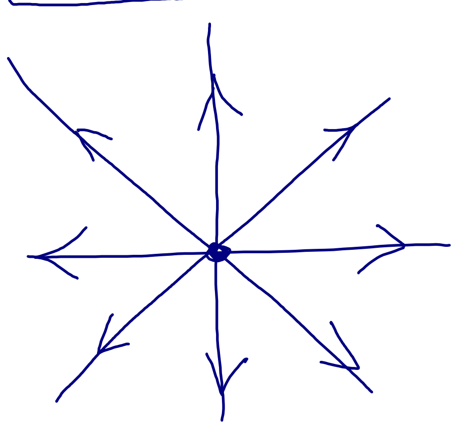
If A is diagonalizable with a double eigenvalue λ then (see §6.2) we have $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

General solution to $\vec{y}' = A\vec{y}$ is
$\vec{y}(t) = e^{\lambda t}\vec{v}$ where $\vec{v}$ is any constant vector.

This gives the following portraits:

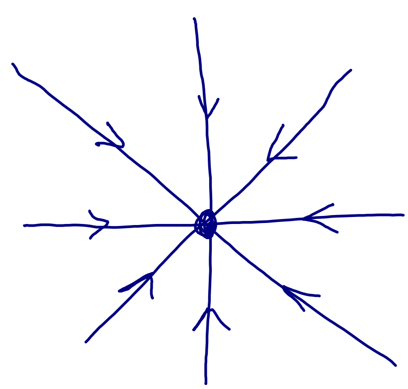
Case 2.1: $\lambda > 0$

STAR SOURCE



Case 2.2: $\lambda < 0$

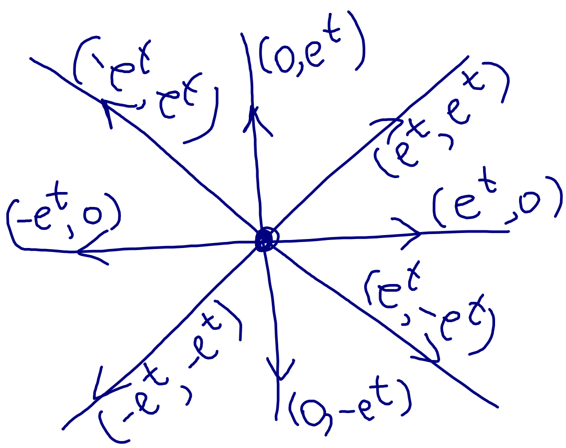
STAR SINK



Example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\lambda = 1$

$$\begin{cases} y_1' = y_1 \\ y_2' = y_2 \end{cases} \qquad \begin{array}{l} y_1 = C_1 e^t \\ y_2 = C_2 e^t \end{array}$$

$\vec{y} = e^t \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ → each trajectory is a straight ray



§6.3.6. Review

$\boxed{\text{TECHNIQUE}}$

To plot the phase portrait of $\vec{y}' = A\vec{y}$,

Step 1: Find eigenvalues & eigenvectors of A

* If one of the eigenvalues $= 0$ or A is not diagonalizable then this is one of the situations which we did not cover in class

Step 2: Use the cases on the next page

2 real eigenvalues	1 eigenvalue λ	2 complex eigenvalues
Plot the special trajectories $\vec{y} = \pm e^{\lambda_1 t}\vec{v}_1,\ \pm e^{\lambda_2 t}\vec{v}_2$ Both e.v. $> 0 \Rightarrow$ $\boxed{\text{Source}}$ Both e.v. $< 0 \Rightarrow$ $\boxed{\text{Sink}}$ e.v. opposite signs $\Rightarrow$ $\boxed{\text{Saddle}}$	$\lambda > 0$ $\Downarrow$ $\boxed{\text{Star source}}$ $\lambda < 0$ $\Downarrow$ $\boxed{\text{Star sink}}$	Look at $\operatorname{tr} A = \lambda_1 + \lambda_2$ $\operatorname{tr} A = 0 \Rightarrow$ $\boxed{\text{Center}}$ $\operatorname{tr} A > 0 \Rightarrow$ $\boxed{\text{spiral source}}$ $\operatorname{tr} A < 0 \Rightarrow$ $\boxed{\text{Spiral sink}}$