§6.2. Solving 2×2 systems using linear algebra

We now find the general solution to the system

$$\boxed{\vec{y}\,'(t) = A\vec{y}(t)} \quad (*)$$

for most 2×2 matrices A, more precisely for those A which are <u>diagonalizable</u> (see §5.2)

The <u>key observation</u> is the following

<u>Fact</u>: if $\vec{v}$ is an eigenvector of A with eigenvalue λ, then

$\vec{y}(t) = e^{\lambda t}\cdot\vec{v}$ is a solution to $(*)$

<u>Proof</u> We compute $\vec{y}\,'(t) = \lambda e^{\lambda t}\cdot\vec{v}$

and $A\vec{y}(t) = e^{\lambda t}\cdot A\vec{v} = e^{\lambda t}\cdot\lambda\vec{v}$. $\square$

This observation leads to the following

<u>Theorem</u> If A is diagonalizable, with $A\vec{v}_1 = \lambda_1\vec{v}_1$, $A\vec{v}_2 = \lambda_2\vec{v}_2$, $\vec{v}_1, \vec{v}_2$ basis of $\mathbb{R}^2$, then the general solution to $(*)$ is given by

$$\boxed{\vec{y}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2}$$

where C_1, C_2 are arbitrary constants.

In other words, using the terminology of §5.3, a basis of the space of solutions to the equation (*) is given by the vector valued functions

$$\vec{y}_1(t) = e^{\lambda_1 t}\vec{v}_1, \quad \vec{y}_2(t) = e^{\lambda_2 t}\vec{v}_2.$$

ALGORITHM for solving a linear 2x2 system of ODEs

TECHNIQUE

Step 1: write the system in vector form

$$\vec{y}\,'(t) = A\vec{y}(t)$$

Step 2: Find a basis of eigenvectors of A:

$$A\vec{v}_1 = \lambda_1 \vec{v}_1, \quad A\vec{v}_2 = \lambda_2 \vec{v}_2$$

Step 3: write the general solution as

$$\vec{y} = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2.$$

Step 4: if λ_1, λ_2 are complex, write the real general solution as $\vec{y} = C_1 \cdot \mathrm{Re}(e^{\lambda_1 t}\vec{v}_1) + C_2 \cdot \mathrm{Im}(e^{\lambda_1 t}\vec{v}_1)$ (see below for an explanation)

Step 5: if solving an initial value problem, use the initial condition to find C_1, C_2.

What if A is not diagonalizable? Can still solve the system (see RS 6 problem 5(b)) but we won't study this here.

Example:

$$\begin{cases} y_1' = y_2 \\ y_2' = y_1 \\ y_1(0) = 1 \\ y_2(0) = 0 \end{cases}$$

PRACTICE

Step 1: $\vec{y} = A\vec{y}$ where $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Step 2: $\operatorname{tr} A = 0$, $\det A = -1$

$$P(\lambda) = \det(\lambda I - A) = \lambda^2 - 1.$$

Eigenvalues $\lambda = \pm 1$, diagonalizable.

Eigenvectors:

$\lambda_1 = 1$	$\lambda_2 = -1$
$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\begin{cases} -x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \end{cases} \to x_1 = 1, x_2 = 1$	$\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{cases} \to x_1 = 1, x_2 = -1$
$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Step 3: General solution

$$\vec{y} = c_1 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 e^{t} + c_2 e^{-t} \\ c_1 e^{t} - c_2 e^{-t} \end{pmatrix}$$

<u>Step 5</u>: plug in the <u>initial condition</u>

$$\begin{pmatrix}1\\0\end{pmatrix} = \vec{y}(0) = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\begin{cases} c_1 + c_2 = 1 & \to 2c_2 = 1 \to c_2 = 1/2 \\ c_1 - c_2 = 0 & \to c_1 = c_2 \longrightarrow c_1 = 1/2 \end{cases}$$

$c_1 = c_2 = 1/2$

$$y(t) = \frac{1}{2}e^{t}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}e^{-t}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}\frac{e^t+e^{-t}}{2}\\ \frac{e^t-e^{-t}}{2}\end{pmatrix}$$

<u>Example with complex eigenvalues</u>:

PRACTICE

$$\begin{cases} y_1' = y_2 \\ y_2' = -y_1 \end{cases}$$

<u>Step 1</u>: $\vec{y}' = A\vec{y}$ where $A = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$

<u>Step 2</u>: eigenvalues of A are $i, -i$

Eigenvectors computed in §5.2:

$A\vec{v}_1 = i\vec{v}_1$, $A\vec{v}_2 = -i\vec{v}_2$ where $\vec{v}_1 = \begin{pmatrix}1\\i\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}1\\-i\end{pmatrix}$

<u>Step 3</u>: Complex general solution

$$\vec{y}(t) = c_1 e^{it}\begin{pmatrix}1\\i\end{pmatrix} + c_2 e^{-it}\begin{pmatrix}1\\-i\end{pmatrix}$$

Complex basis of solutions $e^{it}\begin{pmatrix}1\\i\end{pmatrix}$, $e^{-it}\begin{pmatrix}1\\-i\end{pmatrix}$

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Step 4: to get a real basis, take the Re & Im parts of one of the complex basis elements

$$\mathrm{Re}\left(e^{it}\begin{pmatrix}1\\i\end{pmatrix}\right) = \mathrm{Re}\begin{pmatrix}\cos t + i\sin t\\ i(\cos t + i\sin t)\end{pmatrix} = \begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}$$

$$\mathrm{Im}\left(e^{it}\begin{pmatrix}1\\i\end{pmatrix}\right) = \mathrm{Im}\begin{pmatrix}\cos t + i\sin t\\ i(\cos t + i\sin t)\end{pmatrix} = \begin{pmatrix}\sin t\\ \cos t\end{pmatrix}$$

General solution:

$$\vec{y}(t) = C_1\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + C_2\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}$$

What if we instead took Re, Im of $e^{-it}\begin{pmatrix}1\\-i\end{pmatrix}$ in Step 4? Would arrive to essentially the same real basis:

$$\mathrm{Re}\left(e^{-it}\begin{pmatrix}1\\-i\end{pmatrix}\right) = \begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}, \quad \mathrm{Im}\left(e^{-it}\begin{pmatrix}1\\-i\end{pmatrix}\right) = -\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}.$$

If A has complex (non-real) eigenvalues, you only need to compute one of the eigenvectors. TECHNIQUE