§6. 2×2 linear systems of ODEs

§6.1. Basic properties

In this chapter we study first order linear homogeneous constant coefficient 2×2 systems of ODEs.

$\boxed{\text{THEORY}}$

These have the form

$$\begin{cases} y_1'(t) = a\,y_1(t) + b\,y_2(t) \\ y_2'(t) = c\,y_1(t) + d\,y_2(t) \end{cases} \qquad (*)$$

where $(y_1(t), y_2(t))$ is the unknown pair of functions and a, b, c, d are given real constants.

We will be looking for the general solution as well as solving the initial value problem, with initial conditions of the form

$$\begin{cases} y_1(t_0) = A_1 \\ y_2(t_0) = A_2 \end{cases}$$

where t_0, A_1, A_2 are given numbers

Example 1:

$$\begin{cases} y_1' = y_1 \\ y_2' = y_2 \\ y_1(0) = 1 \\ y_2(0) = -1 \end{cases}$$

Here y_1, y_2 do not interact, so we get the general solution

$$\begin{cases} y_1(t) = C_1 e^t \\ y_2(t) = C_2 e^t \end{cases}$$ for arbitrary constants C_1, C_2

and the solution to the initial value problem is

$$\begin{cases} y_1(t) = e^t \\ y_2(t) = -e^t \end{cases}, \text{ i.e. } (y_1(t), y_2(t)) = (e^t, -e^t)$$

Example 2:

$$\begin{cases} y_1' = y_2 \\ y_2' = -y_1 \end{cases}$$

This system is <u>coupled</u> since y_1' features y_2 (and y_2' features y_1)

The general solution (found using the algorithm of §6.2) is

$$\begin{cases} y_1 = C_1 \cos t + C_2 \sin t \\ y_2 = -C_1 \sin t + C_2 \cos t \end{cases}$$

Motivation: the 2×2 linear systems we work with here are a simple case of more general n×n nonlinear systems of ODEs which are used to model behavior of various time-dependent systems.

For example, the motion of a planet around the Sun is modeled by a system of 6 ODEs with 6 unknown functions:
$y_1(t), y_2(t), y_3(t)$ coordinates of the planet
$v_1(t), v_2(t), v_3(t)$ coordinates of its velocity
[Newton 1684]

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§6.1.1. Vector notation

Define the vector-valued function $\vec{y}: \mathbb{R} \to \mathbb{R}^2$ by

THEORY

$$\vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

Then the system (*) can be rewritten as

$$\vec{y}'(t) = A\vec{y}(t) \text{ where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix.

Example:

PRACTICE TECHNIQUE

Write the system

$$\begin{cases} y_1' = y_2 \\ y_2' = -y_1 \end{cases}$$

in the vector form

Solution:

$$\begin{cases} y_1' = 0 \cdot y_1 + 1 \cdot y_2 \\ y_2' = -1 \cdot y_1 + 0 \cdot y_2 \end{cases}$$

Thus $\vec{y}\,' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{y}$.

Note: the general solution (see above) can be written as

$$\vec{y}(t) = C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

I.e. $\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ form a basis of the solution space.

§6.1.2. Converting 2^{nd} order ODEs into 1^{st} order systems

THEORY
TECHNIQUE

If we have an ODE

$$y'' + a_1 y' + a_0 y = 0 \quad (\star)$$

where a_0, a_1 are constants

then we can convert it to a 2×2 1^{st} order system with the unknown vector-valued function

$$\vec{y}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}.$$

Indeed, if we denote $y_1 := y$, $y_2 := y'$ then

$$\begin{cases} y_1' = y_2 & (\text{i.e. } y' = y') \\ y_2' = -a_0 y_1 - a_1 y_2 & (\text{i.e. } y'' = -a_0 y - a_1 y') \end{cases}$$

which can be written as

$$\vec{y}'(t) = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \vec{y}(t).$$

This is called the <u>companion system</u> of the ODE $(\star)$ and the matrix $\begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}$ is called the <u>companion matrix</u>.

Example: harmonic oscillator

PRACTICE

$$y'' + y = 0$$

If $y_1 = y$, $y_2 = y'$ then

$$\begin{cases} y_1' = y_2 \\ y_2' = y'' = -y = -y_1 \end{cases}$$

The companion system is

$$\vec{y}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{y} \quad \text{where } \vec{y}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}.$$

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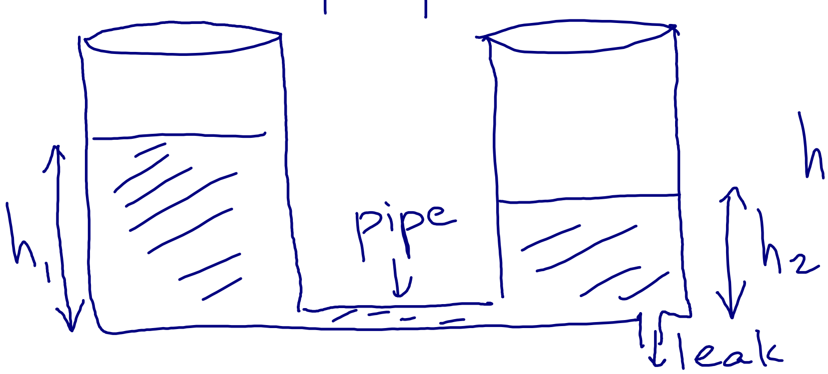
§6.1.3. Example: two connected tanks

Two identical tanks are connected by a pipe. The flow rate in the pipe is proportional to the difference of fluid levels in the tanks. One tank is leaking, with the rate of leakage proportional to its fluid level.



t = time (in minutes)

$h_1(t), h_2(t)$ = heights of the tanks (in meters)

MITx 2.1.6-2.1.7

We get the ODEs

$$\frac{dh_1}{dt} = a(h_2 - h_1)$$

where $a > 0$ is some coefficient (measured in $1/\text{min}$)

This accounts for fluid transfer via the pipe

$$\frac{dh_2}{dt} = a(h_1 - h_2) - bh_2$$

$b \geq 0$ is a coefficient measured in $1/\text{min}$

This accounts for the pipe $(a(h_1 - h_2))$ and the leakage $(-bh_2)$

We got the system of ODEs

$$\begin{cases} h_1' = -ah_1 + ah_2 \\ h_2' = ah_1 - (a+b)h_2 \end{cases}$$

The vector form of this is

$$\vec{h}' = A\vec{h} \quad \text{where } \vec{h} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad A = \begin{pmatrix} -a & a \\ a & -a-b \end{pmatrix}$$