§5.3. Vector spaces

Here we introduce briefly
the abstract concept of a vector space
which in particular lets us study
the spaces of solutions to differential equations
using the methods of linear algebra.

THEORY

Definition A vector space is a set V
with operations
$$\vec{x}, \vec{y} \text{ in } V \longmapsto \vec{x}+\vec{y} \text{ in } V$$
$$\vec{x} \text{ in } V, c \text{ in } \mathbb{R} \text{ (or } \mathbb{C}) \longmapsto c\vec{x} \text{ in } V$$
which satisfy some natural properties which
I do not list here (e.g. $\vec{x}+\vec{y}=\vec{y}+\vec{x}$)

That is, one can add elements of
a vector space and one can
multiply them by scalars

Examples 1. $\mathbb{R}^n$ is a vector space

2. The space $Fns(\mathbb{R})$ of all functions
$f: \mathbb{R} \to \mathbb{R}$ is a vector space

$$(f+g)(t) \overset{\text{def}}{=} f(t)+g(t)$$
$$(cf)(t) \overset{\text{def}}{=} c \cdot f(t)$$

operations are defined pointwise.

For a vector space V, we can define linear combination, span, linear independence as we did for $\mathbb{R}^n$:

- Linear combination of $\vec{x}_1, \dots, \vec{x}_k$ in V has the form $c_1\vec{x}_1 + \dots + c_k\vec{x}_k$ where $c_1, \dots, c_k$ are scalars
- Span $(\vec{x}_1, \dots, \vec{x}_k)$ = the set of all linear combinations of $\vec{x}_1, \dots, \vec{x}_k$
- $\vec{x}_1, \dots, \vec{x}_k$ are linearly independent if

$$c_1\vec{x}_1 + \dots + c_k\vec{x}_k = 0 \implies c_1 = \dots = c_k = 0$$

$\boxed{\text{THEORY}}$

Basis

We say that $\vec{x}_1, \dots, \vec{x}_k$ (in V) are a basis of V, if:

1. $\vec{x}_1, \dots, \vec{x}_k$ are linearly independent, and
2. Span $(\vec{x}_1, \dots, \vec{x}_k) = V$.

In this case each $\vec{x}$ in V can be written as $\vec{x} = c_1\vec{x}_1 + \dots + c_k\vec{x}_k$ for a unique choice of $c_1, \dots, c_k$, called the coordinates of $\vec{x}$ in the basis $\vec{x}_1, \dots, \vec{x}_k$.

If V has a basis $\vec{x}_1, \dots, \vec{x}_k$ we say that V is <u>finite dimensional</u> and call k the <u>dimension</u> of V.

If V has no (finite) basis then we say that V is <u>infinite dimensional</u>

<u>Examples</u> The space of functions $Fns(\mathbb{R})$ is infinite dimensional; $\mathbb{R}^n$ is n-dimensional

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THEORY

Linear maps on vector spaces

If V, W are vector spaces then a map $T: V \to W$ is called <u>linear</u> if

1. $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ for all $\vec{x}, \vec{y}$ in V

and

2. $T(c\vec{x}) = c\,T(\vec{x})$ for all $\vec{x}$ in V, c scalar

<u>Examples</u>: 1. $T_A(\vec{x}) = A \cdot \vec{x}$ is a linear map $\mathbb{R}^m \to \mathbb{R}^n$ for any $n \times m$ matrix A.

2. $D: Fns(\mathbb{R}) \to Fns(\mathbb{R})$ is the linear map defined by $D(f) = f'$

(not actually true because not every function can be differentiated but we ignore this problem here)

$f(t) = e^{\lambda t}$ is an eigenfunction of D with eigenvalue λ: $\boxed{D(f) = \lambda \cdot f}$ since $f' = \lambda e^{\lambda t}$

(3) $P(D): \mathrm{Fns}(\mathbb{R}) \to \mathrm{Fns}(\mathbb{R})$

$P(D)y = a_k y^{(k)} + \cdots + a_1 y' + a_0 y$, $P(z) = a_k z^k + \cdots + a_0$

(4) $T: \mathrm{Fns}(\mathbb{R}) \to \mathbb{R}^2$ is a linear map where

$$T(f) \overset{\text{def}}{=} \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}$$

Null space

If $T: V \to W$ is a linear map then we define the null space of T to be the set of all solutions $\vec{v}$ to the equation

$$\boxed{T(\vec{v}) = 0.}$$

The null space of T is a subspace of V, namely

(1) $\vec{v}_1, \vec{v}_2$ in the null space $\Rightarrow$ $\vec{v}_1 + \vec{v}_2$ in the null space

and

(2) $\vec{v}$ in the null space $\Rightarrow$ $c\vec{v}$ in the null space for all scalars c.

Thus the null space of any linear map T forms a vector space

Example:

Let S = set of all solutions to the ODE

$$y'' + y = 0.$$

Then S = null space of the operator
$T = P(D)$ where $P(z) = z^2 + 1$
$Ty = y'' + y$.
We know that (allowing complex valued solutions)
S = the set of all functions of the form
$$y(t) = c_1 e^{it} + c_2 e^{-it}$$
Thus S is 2-dimensional and
e^{it}, e^{-it} form a basis of S.
Another basis of S is given by $\cos t, \sin t$

<u>General fact</u>: the space of all solutions
to a k-th order linear homogeneous ODE
(e.g. $P(D)y = 0$, $P = a_k z^k + \cdots$, $a_k \neq 0$)
is a k-dimensional vector space.
For constant coefficient equations $P(D)y = 0$
the algorithm in §4.4 gives a <u>basis</u>
of this space