§5.2. Matrices

An $n \times m$ <u>matrix</u> is a table of (real) numbers with n rows & m columns

It is denoted $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$

a_{jk} = element in j-th row, k-th column

<u>Example:</u> $A = \begin{pmatrix} 1 & 3 & 7 \\ 2 & 4 & -3 \end{pmatrix}$ is a 2×3 matrix

If $A = (a_{jk})$ then $a_{11} = 1$, $a_{12} = 3$, $a_{13} = 7$
$a_{21} = 2$, $a_{22} = 4$, $a_{23} = -3$

Multiplying a matrix by a vector

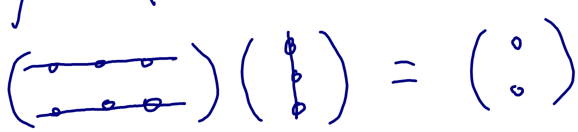
$\boxed{\text{TECHNIQUE}}$

If A is an $n \times m$ matrix and $\vec{x}$ is a vector of length m, then $A\vec{x}$ will be a vector of length n.

To compute $A\vec{x}$:
j-th entry of $A\vec{x}$ is obtained by multiplying the entries of j-th row of A by the entries of $\vec{x}$ and taking the sum

<u>Examples:</u>

1. $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \cdot (-1) + 2 \cdot 2 + 1 \cdot 3 \\ (-1)(-1) + 0 \cdot 2 + 2 \cdot 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$

Helpful drawing: 

2 $\begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -2 & +0 \\ -1 & +2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$

§5.2.1. Matrices as linear maps

Let A be an $n \times m$ matrix.
Consider the map $T_A : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ defined by

(*) $\boxed{T_A(\vec{x}) = A \cdot \vec{x}}$ for all $\vec{x}$ in $\mathbb{R}^m$

Then T_A is a linear map, i.e.

(1) $\boxed{T_A(\vec{x}+\vec{y}) = T_A(\vec{x}) + T_A(\vec{y})}$ for all $\vec{x}, \vec{y}$ in $\mathbb{R}^m$
and
(2) $\boxed{T_A(c\vec{x}) = c\,T_A(\vec{x})}$ for all $\vec{x}$ in $\mathbb{R}^m$, c in $\mathbb{R}$.

(That is, $A(\vec{x}+\vec{y}) = A \cdot \vec{x} + A \cdot \vec{y}$ and $A \cdot c\vec{x} = c \cdot A\vec{x}$)

Theorem Assume that $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation (i.e. it satisfies (1) + (2) above). Then $T = T_A$ for some matrix A.

THEORY

That is, all linear transformations are given by matrices.

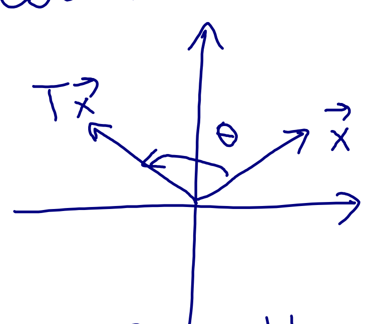
Given a linear map $T: \mathbb{R}^m \to \mathbb{R}^n$, how to find the matrix A such that $T(\vec{x}) = A \cdot \vec{x}$?

Use the <u>canonical basis</u> $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \ldots$

$A = (a_{jk})$ where a_{jk} (the element in j-th row & k-th column)

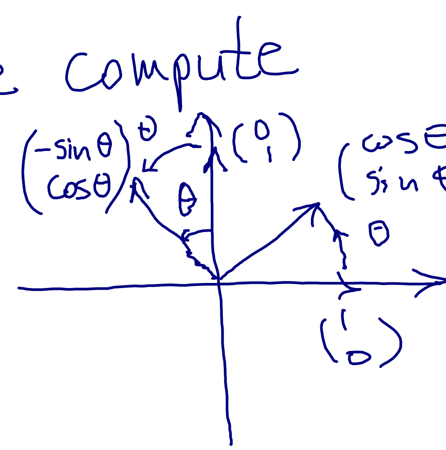
is equal to the j-th entry of the vector $T(\vec{e}_k)$ $\boxed{\text{TECHNIQUE}}$

<u>Example</u> $T: \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation counterclockwise by angle θ. $\boxed{\text{PRACTICE}}$



It is known that T is linear.

To find the matrix of T, we compute



$$T(\vec{e}_1) = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$T(\vec{e}_2) = T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

So $T(\vec{x}) = A \cdot \vec{x}$ where

$$A = \left(T\begin{pmatrix} 1 \\ 0 \end{pmatrix} \;\; T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

§5.2.2. Operations with matrices

Identity matrix of size $n\times n$:

$$I=\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \;\; 1 \end{pmatrix} \quad \text{e.g. } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ etc.}$$

The corresponding linear transformation is the identity map: for $\vec{x}$ in $\mathbb{R}^n$,

$$T_I(\vec{x}) \stackrel{\text{def}}{=} \boxed{I\cdot\vec{x} = \vec{x}.}$$

THEORY

TECHNIQUE

Matrix addition:

If A, B are matrices of the same size then $A+B$ is defined entry-wise

e.g. $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

Multiplication by scalars:

As for vectors, this is done entry-wise:

e.g. $$2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

Vectors as matrices:

We view any vector in $\mathbb{R}^n$ as a column vector, i.e. an $n\times 1$ matrix.

Matrix multiplication:

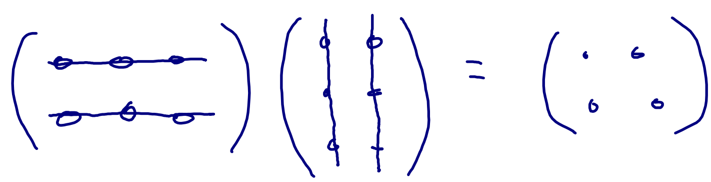
if A is an $n \times m$ matrix
and B is an $m \times k$ matrix ← need #(columns of A) = #(rows of B)
then AB is an $n \times k$ matrix

The j-th column of AB is obtained by multiplying A by the j-th column of B (using multiplication of a matrix by a vector)

Example: $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 0 \end{pmatrix}$

$$AB = \begin{pmatrix} 1\cdot 1 + 0\cdot(-1) + 1\cdot 1 & 1\cdot 2 + 0\cdot 1 + 1\cdot 0 \\ 2\cdot 1 + (-1)\cdot(-1) + 0\cdot 1 & 2\cdot 2 + (-1)\cdot 1 + 0\cdot 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

Helpful drawing:



$$BA = \begin{pmatrix} 5 & -2 & 1 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$ (check at home if you have time)

Note: we have $\boxed{(AB)C = A(BC)}$ where A, B, C are any matrices and $\boxed{I \cdot A = A \cdot I = A}$ but in general $\boxed{A \cdot B \neq B \cdot A}$ (not even same size...)

What does matrix multiplication mean for the corresponding linear maps T_A? We have

$$T_{AB}(\vec{x}) = T_A(T_B(\vec{x})),$$

that is $(AB)\cdot\vec{x} = A\cdot(B\cdot\vec{x})$.

THEORY

$$\boxed{\begin{array}{c}\text{Multiplication of matrices} \\ \| \\ \text{Composition of linear maps}\end{array}}$$

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§5.2.3. Linear systems & null space

Here we look at a matrix equation

$\boxed{A\vec{x} = \vec{b}}$ where

THEORY

A is an $n \times m$ matrix
$\vec{x}$ is an unknown vector of length m
$\vec{b}$ is a given vector of length n

Note that this equation is equivalent to a system of n linear equations on the entries of $\vec{x}$

Example Write the matrix eqn.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \vec{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$ as a system of linear equations

TECHNIQUE PRACTICE

Solution Step 1: write $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Note: $\vec{x}$ has 3 entries because the matrix has 3 columns

Step 2: Use matrix-vector multiplication

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 + x_3 \\ x_1 - x_2 + 2x_3 \end{pmatrix}$$

Step 3: Write the system

$$\begin{cases} x_2 + x_3 = 3 \\ x_1 - x_2 + 2x_3 = 4 \end{cases}$$

Now we focus on the case $\vec{b} = 0$:

Definition Let A be a matrix. The null space of A is the set of all solutions $\vec{x}$ to the homogeneous linear equation $\boxed{A\vec{x} = \vec{0}}$

The null space of any matrix A contains the zero vector, as $A\cdot\vec{0}=\vec{0}$.
To see if it contains any other vectors we study the corresponding system of linear equations:

$\boxed{\begin{array}{c}\text{TECHNIQUE}\\ \hline \text{PRACTICE}\end{array}}$

<u>Exercise</u> Do the following matrices have a nonzero vector in their nullspace? If so, find such a vector.

(a) $A=\begin{pmatrix}1&2\\3&6\end{pmatrix}$ (b) $A=\begin{pmatrix}1&2\\2&3\end{pmatrix}$

<u>Solution</u> (a) <u>Step 1:</u> write as a linear system

$$A\vec{x}=\vec{0} \iff \begin{cases} x_1+2x_2=0\\ 3x_1+6x_2=0\end{cases}$$

<u>Step 2:</u> analyze the system
(more systematic procedure in Linear Algebra II later
$x_1+2x_2=0 \iff x_1=-2x_2$. Substitute into the 2^{nd} eqn.:
$3x_1+6x_2=0 \iff -6x_2+6x_2=0 \iff 0=0$, always true
So we just need to find a nonzero solution to $x_1+2x_2=0$. Can take $x_2=1$, $x_1=-2$
Get $\vec{x}=\begin{pmatrix}-2\\1\end{pmatrix}$ a nonzero vector in the nullspace of A

(b) Step 1: get $\begin{cases} x_1 + 2x_2 = 0 \\ 2x_1 + 3x_2 = 0 \end{cases}$

Step 2: use the 1^{st} eqn. to write
$x_1 = -2x_2$. Substitute into the 2^{nd} eqn.:
$-4x_2 + 3x_2 = 0 \iff x_2 = 0$
Then also $x_1 = 0$.
The null space has no nonzero vectors.

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§5.2.4. 2×2 matrices: determinant

Assume that A is a 2×2 matrix.
When does A have a nonzero vector in its nullspace?

THEORY

Theorem Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix.
Define its determinant
$\boxed{\det A = a \cdot d - b \cdot c.}$
Then the equation $A\vec{x} = \vec{0}$ has a nonzero solution $\vec{x} \iff \det A = 0.$

Proof (optional)
We restrict to the case $a \neq 0$.

We have $A\vec{x}=\vec{0} \iff \begin{cases} ax_1+bx_2=0 \\ cx_1+dx_2=0 \end{cases}$

Use the 1^{st} eqn. to write $\boxed{x_1 = -\frac{b}{a}x_2}$

Substitute into the 2^{nd} eqn. to get

$\left(d-\frac{bc}{a}\right)x_2=0.$ (*) Note: $d-\frac{bc}{a} = \frac{\det A}{a}$.

If <u>det $A=0$</u> then (*) is always satisfied.

So $x_2=1$, $x_1=-\frac{b}{a}$ gives a vector $\vec{x}=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in the null space of A

If <u>det $A\neq 0$</u> then $(*) \Rightarrow x_2=0 \Rightarrow x_1=0$.

So there is no nonzero vector in the null space. □

<u>Examples</u>

① $A=\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \Rightarrow \det A = 1\cdot 6 - 2\cdot 3 = 0$

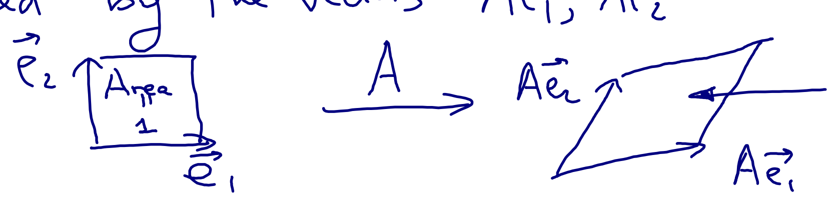
The eqn. $A\vec{x}=0$ has a nonzero solution

② $A=\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \Rightarrow \det A = 1\cdot 3 - 2\cdot 2 = -1 \neq 0$

The eqn. $A\vec{x}=0$ has no nonzero solutions.

<u>Geometric interpretation (optional)</u>: if $\vec{e}_1=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{e}_2=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

then $\det A =$ signed <u>area</u> of the parallelogram spanned by the vectors $A\vec{e}_1$, $A\vec{e}_2$



Area = det A.

§5.2.5. 2×2 matrices: eigenvalues & eigenvectors

Assume that A is a 2×2 matrix.

We study the eigen-equation

THEORY

(*) $\boxed{A\vec{x} = \lambda\vec{x}}$

where $\vec{x}$ is a vector (possibly with complex entries) and λ is a number (possibly complex)

Definition: We say λ is an eigenvalue of A if (*) has a nonzero solution $\vec{x}$.

If λ is an eigenvalue, then we call the nonzero solutions $\vec{x}$ to (*) eigenvectors of A associated to λ.

Our goal will be to find a basis of $\mathbb{R}^2$ (actually, of $\mathbb{C}^2$ because $\vec{x}$ might be complex) consisting of eigenvalues of A:

$$\boxed{A\vec{x}_1 = \lambda_1\vec{x}_1, \quad A\vec{x}_2 = \lambda_2\vec{x}_2,}$$

$\vec{x}_1, \vec{x}_2$ a basis (i.e. linearly independent)

Why? Because in this basis the matrix A will have a simple form & it will help us solve systems of 2 ODEs

Definition We call A *diagonalizable* if there exists a basis of eigenvalues of A

(Why diagonalizable? Because the matrix of the linear map T_A in the basis $\vec{x}_1, \vec{x}_2$ will be the diagonal matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ ...)

Finding eigenvalues

THEORY

We rewrite the eigen-equation $(*)$ as

$$(A-\lambda I)\vec{x} = 0 \quad \text{where } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \lambda I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

From §5.2.4 we know that

λ is an eigenvalue $\iff$ $\lambda I - A$ has a nonzero vector in its nullspace $\iff$

$$\iff \boxed{\det(\lambda I - A) = 0} \qquad (**)$$

We call $(**)$ the characteristic equation of A

The function $\boxed{p(\lambda) = \det(\lambda I - A)}$

is a quadratic polynomial called the characteristic polynomial of A

To recap:

$$\boxed{\text{Eigenvalues of } A = \text{Roots of } p(\lambda)}$$

Example: $A = \begin{pmatrix} 2 & 2 \\ -2 & -3 \end{pmatrix}$

Compute $P(\lambda) = \det(\lambda I - A)$

$= \det \begin{pmatrix} \lambda - 2 & -2 \\ 2 & \lambda + 3 \end{pmatrix} = (\lambda - 2)(\lambda + 3) - (-2)\cdot 2$

$= \lambda^2 + \lambda - 2.$

Roots $\lambda = \dfrac{-1 \pm \sqrt{1 + 2\cdot 4}}{2} = \dfrac{-1 \pm 3}{2} = 1; -2$

Useful formula: if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\boxed{\det(\lambda I - A) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A}$$

where $\operatorname{tr} A \stackrel{\text{def}}{=} a + d$ (called the trace of A)

$\det A = ad - bc$

Example above: $\operatorname{tr} A = 2 - 3 = -1$

$\det A = 2\cdot(-3) - 2\cdot(-2) = -2$

$P(\lambda) = \lambda^2 + \lambda - 2.$

How to find eigenvectors corresponding to the eigenvalues?

Reduce to solving systems of linear equations.

Example: still $A = \begin{pmatrix} 2 & 2 \\ -2 & -3 \end{pmatrix}$

Eigenvalues $1, -2$ (computed before)

PRACTICE TECHNIQUE

Eigenvector with eigenvalue 1:

looking for nonzero $\vec{x}$ such that
$(A - I)\vec{x} = 0$. Compute $A - I = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$

Write as linear system:

$\begin{cases} x_1 + 2x_2 = 0 \\ -2x_1 - 4x_2 = 0 \end{cases}$ $\longrightarrow$ Solution $x_2 = 1,\ x_1 = -2$

$\boxed{\text{Eigenvector } \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$

(Check: $A\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \ldots$)

Eigenvector with eigenvalue -2:

$(A + 2I)\vec{x} = 0$, $A + 2I = \begin{pmatrix} 4 & 2 \\ -2 & -1 \end{pmatrix}$

$\begin{cases} 4x_1 + 2x_2 = 0 \\ -2x_1 - x_2 = 0 \end{cases}$ $\rightarrow$ can take $x_1 = 1,\ x_2 = -2$

$\boxed{\text{Eigenvector } \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$

(Check: $A\begin{pmatrix} 1 \\ -2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -2 \end{pmatrix}$)

ALGORITHM for finding a basis of eigenvectors for a 2x2 matrix A

TECHNIQUE

Step 1: Compute the characteristic polynomial $p(\lambda)$.

Can use the formulas $p(\lambda) = \det(\lambda I - A)$

or $p(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A$

(the second formula is faster to use)

Step 2: Find the roots of $p(\lambda)$.

These are the eigenvalues of A.

Step 3(a): if there are 2 distinct eigenvalues λ_1, λ_2 (real or complex)

find eigenvectors $\vec{x}_1, \vec{x}_2$: nonzero solutions to $(A - \lambda_1 I)\vec{x}_1 = 0$, $(A - \lambda_2 I)\vec{x}_2 = 0$.

It is guaranteed that $\vec{x}_1, \vec{x}_2$ will form a basis and thus A is diagonalizable

Step 3(b): if there is only one eigenvalue λ

then either $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

in which case A is diagonalizable because $A\vec{x} = \lambda\vec{x}$ for all $\vec{x}$ (e.g. can take $\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$)

or $A \neq \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

in which case A is not diagonalizable.

Example: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\operatorname{tr} A = 0$, $\det A = 1$

$p(\lambda) = \lambda^2 + 1$

Eigenvalues $\boxed{\lambda_1 = i, \ \lambda_2 = -i}$; diagonalizable

$\lambda = i$: $(A - iI)\vec{x} = 0$; $A - iI = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}$

$\begin{cases} -ix_1 + x_2 = 0 \\ -x_1 - ix_2 = 0 \end{cases}$ Put $x_1 = 1$, then $x_2 = i$

Eigenvector $\boxed{\vec{x}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix},}$ $A\vec{x}_1 = \lambda_1 \vec{x}_1$

$\lambda = -i$: $(A + iI)\vec{x} = 0$; $A + iI = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$

$\begin{cases} ix_1 + x_2 = 0 \\ -x_1 + ix_2 = 0 \end{cases}$ Put $x_1 = 1$, then $x_2 = -i$

Eigenvector $\boxed{\vec{x}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix},}$ $A\vec{x}_2 = \lambda_2 \vec{x}_2$

Example: $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

$\operatorname{tr} A = 4$, $\det A = 4$

$p(\lambda) = \lambda^2 - 4\lambda + 4$

$\boxed{\text{Eigenvalue } \lambda = 2 \text{ (multiplicity 2)}}$

$A \neq \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow$ not diagonalizable.

Why not? Look for eigenvectors:

$(A - 2I)\vec{x} = 0$, $A - 2I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\begin{cases} x_2 = 0 \\ 0 = 0 \end{cases} \rightarrow$ eigenvectors have the form

$\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ where $x_1 \neq 0$

These are not enough to span $\mathbb{R}^2$.