§5. LINEAR ALGEBRA

This chapter will be used for solving systems of differential equations (which is done in later chapters).

Here we will learn about <u>vectors</u>, <u>matrices</u>, <u>linear combinations</u>, <u>linear independence</u>, <u>span</u>, and <u>basis</u>. We will also learn the abstract notion of a <u>vector space</u> which will let us treat <u>functions</u> like <u>vectors</u>.

The latter is a fundamentally important idea which is used all the time by solving differential equations numerically:

<u>infinite dimensional objects</u> (functions, differential operators)

replaced by

<u>finite dimensional objects</u> (vectors, matrices)

which a computer can deal with.

§5.1. Vectors

THEORY

Denote by $\mathbb{R}$ the set of all real numbers. In this section we will use the term "scalar" to mean a real number.
(One can instead use $\mathbb{C}$, the set of all complex numbers, and we will sometimes do it later in the course.)

Definition A vector of length n is an ordered n-tuple of numbers, denoted $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. Here $x_1, \dots, x_n$ are real numbers called components or entries of the vector $\vec{x}$.

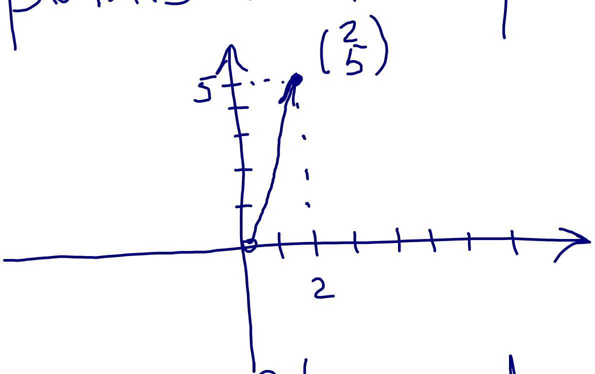
We denote by $\mathbb{R}^n$ the set of all vectors of length n.

Examples: $\begin{pmatrix} 1 \\ 7 \\ 9 \end{pmatrix}$ is in $\mathbb{R}^3$

$\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ is in $\mathbb{R}^2$.

Graphical representation of vectors

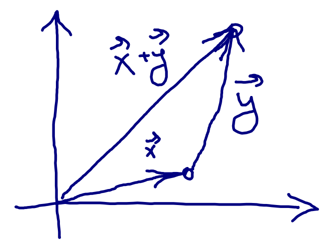
Elements of $\mathbb{R}^2$ can be pictured as points on the plane: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longleftrightarrow$ point with coordinates (x_1, x_2)



We also often draw $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ as an arrow from the origin $(0,0)$ to the point (x_1, x_2).

Elements of $\mathbb{R}^3$ can similarly be pictured as points (or arrows) in space.

Operations on vectors



Addition: component-wise

e.g. $\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1+2 \\ 3+4 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$

We can only add vectors of same length, e.g. $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ is not allowed

Multiplication by scalars:

if c is a number and

$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is a vector then

$$c \cdot \vec{x} = \begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix}$$

e.g. $3\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 \\ 3 \cdot 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

Zero vector: $\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

(how many entries it has will be clear from the context)

Linear combinations, independence, span, basis

- If $\vec{x}_1, \dots, \vec{x}_k$ are vectors of same length then a $\boxed{\text{linear combination}}$ of $\vec{x}_1, \dots, \vec{x}_k$ is a vector of the form $c_1\vec{x}_1 + \cdots + c_k\vec{x}_k$

Example: $\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{x}_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$2\vec{x}_1 - \vec{x}_2 + \vec{x}_3 = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

In general

$$c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = \begin{pmatrix} c_1 + c_2 + 2c_3 \\ c_2 + c_3 \end{pmatrix}$$

- The $\boxed{\text{span}}$ of $\vec{x}_1, \ldots, \vec{x}_k$ is the set of all vectors which can be written as a linear combination of $\vec{x}_1, \ldots, \vec{x}_k$

Examples ① $\text{Span}\left(\begin{pmatrix}1\\-1\end{pmatrix}\right)$

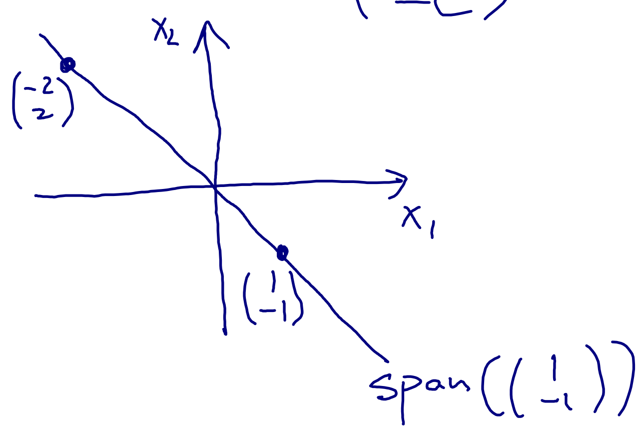
= the set of all vectors of the form $\begin{pmatrix}c_1\\-c_1\end{pmatrix}$

= the set of all vectors $\begin{pmatrix}x_1\\x_2\end{pmatrix}$ such that $x_1+x_2=0$

② $\text{span}\left(\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}-2\\2\end{pmatrix}\right) = \text{span}\left(\begin{pmatrix}1\\-1\end{pmatrix}\right)$

Indeed, each vector $\begin{pmatrix}c_1-2c_2\\-c_1+2c_2\end{pmatrix}$ can be written as $\begin{pmatrix}c\\-c\end{pmatrix}$

where $c = c_1 - 2c_2$



Note:

- in $\mathbb{R}^2$, the span of any set of vectors is either $\vec{0}$, a line through $\vec{0}$, or the entire $\mathbb{R}^2$.
- in $\mathbb{R}^3$, the span could be $\vec{0}$, a line through $\vec{0}$, a plane through $\vec{0}$, or the entire $\mathbb{R}^3$

3. $\mathrm{Span}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right) = \mathbb{R}^2$.

That is, any vector in $\mathbb{R}^2$ written as a linear combination of $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$.

Indeed, if $\vec{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ then $\vec{x} = x_1\begin{pmatrix}1\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\end{pmatrix}$

- The vectors $\vec{x}_1, \dots, \vec{x}_k$ are called $\boxed{\text{linearly independent}}$, if for each $c_1, \dots, c_k$, if $c_1\vec{x}_1 + \dots + c_k\vec{x}_k = \vec{0}$
  then $c_1 = \dots = c_k = 0$.
  (No nontrivial linear combination is equal to $\vec{0}$)

Examples:

$\boxed{\text{PRACTICE TECHNIQUE}}$

1. $\vec{a} = \begin{pmatrix}1\\-1\end{pmatrix}$, $\vec{b} = \begin{pmatrix}-2\\2\end{pmatrix}$ are
<u>linearly dependent</u>: $2\vec{a} + \vec{b} = 0$

2. $\vec{a} = \begin{pmatrix}1\\-1\end{pmatrix}$, $\vec{b} = \begin{pmatrix}1\\1\end{pmatrix}$ are
<u>linearly independent</u>: indeed, assume

$$\vec{0} = c_1\vec{a} + c_2\vec{b} = \begin{pmatrix}c_1 + c_2\\-c_1 + c_2\end{pmatrix}. \text{ Then}$$

$$\begin{cases} c_1 + c_2 = 0 \\ -c_1 + c_2 = 0 \Rightarrow c_1 = c_2 \end{cases} \Rightarrow 2c_1 = 0 \Rightarrow c_1 = 0,\ c_2 = 0.$$

- We call the vectors $\vec{x}_1, \dots, \vec{x}_k$ a $\boxed{\text{basis}}$ of $\mathbb{R}^n$, if

1. $\vec{x}_1, \dots, \vec{x}_k$ are linearly independent, and
2. $\text{Span}(\vec{x}_1, \dots, \vec{x}_k) = \mathbb{R}^n$,

i.e. each vector in $\mathbb{R}^n$ can be written as a linear combination of $\vec{x}_1, \dots, \vec{x}_k$.

Example: $\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a basis of $\mathbb{R}^2$.

(follows from linear independence & one of the theorems below)

The reason we care about basis is

Theorem. If $\vec{x}_1, \dots, \vec{x}_k$ is a basis of $\mathbb{R}^n$ then for each $\vec{x}$ in $\mathbb{R}^n$, there exist unique $c_1, \dots, c_k$ such that

$$\boxed{\vec{x} = c_1 \vec{x}_1 + \dots + c_k \vec{x}_k}$$

In other words, every vector in $\mathbb{R}^n$ can be written in a unique way as a linear combination of $\vec{x}_1, \dots, \vec{x}_k$.

The numbers $c_1, \dots, c_k$ are called the coordinates of $\vec{x}$ in the basis $\vec{x}_1, \dots, \vec{x}_k$

Example Find the coordinates of the vector $\begin{pmatrix}0\\1\end{pmatrix}$ in the basis $\vec{x}_1 = \begin{pmatrix}1\\-1\end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix}1\\1\end{pmatrix}$.

$\boxed{\text{PRACTICE}}$
$\boxed{\text{TECHNIQUE}}$

Solution We write $\begin{pmatrix}0\\1\end{pmatrix} = c_1\vec{x}_1 + c_2\vec{x}_2$ where c_1, c_2 are unknown coefficients.

We need

$$\begin{pmatrix}0\\1\end{pmatrix} = c_1\begin{pmatrix}1\\-1\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}c_1+c_2\\-c_1+c_2\end{pmatrix}$$

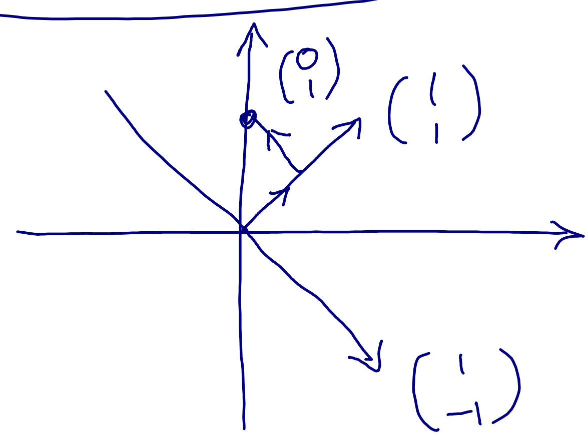
Thus we need c_1, c_2 to satisfy the system of linear equations

$$\begin{cases} c_1 + c_2 = 0 \\ -c_1 + c_2 = 1. \end{cases}$$

To solve this, we use the first equation to set $c_2 = -c_1$, then the second equation gives $-2c_1 = 1 \Rightarrow c_1 = -\frac{1}{2}$, $c_2 = \frac{1}{2}$

That is,

$$\begin{pmatrix}0\\1\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}$$



Theorem (properties of basis in $\mathbb{R}^n$)

1. Every basis of $\mathbb{R}^n$ has exactly n vectors. (e.g. cannot have a basis of $\mathbb{R}^3$ consisting of 2 vectors)
2. If $\vec{x}_1, \dots, \vec{x}_n$ are n linearly independent vectors in $\mathbb{R}^n$, then they form a basis.
3. If $\vec{x}_1, \dots, \vec{x}_n$ are n vectors in $\mathbb{R}^n$ and $\mathrm{Span}(\vec{x}_1, \dots, \vec{x}_n) = \mathbb{R}^n$ then the vectors $\vec{x}_1, \dots, \vec{x}_n$ form a basis.

Part 1 of the above theorem is a special case of

Theorem 1. Every linearly independent set in $\mathbb{R}^n$ has no more than n vectors

2. Every set of vectors which span $\mathbb{R}^n$ has at least n vectors.

Example: $\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{x}_3 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

has to be linearly dependent because it has 3 vectors in $\mathbb{R}^2$ (and $3 > 2$)

How can we find explicit c_1, c_2, c_3 such that $c_1\begin{pmatrix}1\\0\end{pmatrix}+c_2\begin{pmatrix}1\\1\end{pmatrix}+c_3\begin{pmatrix}1\\2\end{pmatrix}=\vec{0}$?

We need $\begin{cases} c_1 + c_2 + c_3 = 0 \\ \quad\;\; c_2 + 2c_3 = 0 \end{cases}$

$\boxed{\begin{array}{c}\text{PRACTICE}\\ \hline \text{TECHNIQUE}\end{array}}$

2 equations for 3 unknowns...

Let's fix one of the coefficients:

Say $c_1 = 1$ → then we get $\begin{cases} c_2 + c_3 = -1 \\ c_2 + 2c_3 = 0 \end{cases}$

$c_2 = -2c_3$, $-1 = c_2 + c_3 = -c_3 \Rightarrow c_3 = 1,\ c_2 = -2,\ c_1 = 1$

$$\begin{pmatrix}1\\0\end{pmatrix} - 2\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} = \vec{0}$$

Later we study <u>Gauss elimination</u> which gives an <u>algorithm</u> for solving systems of linear equations.

<u>Canonical basis of $\mathbb{R}^n$</u>:

$$\vec{e}_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix},\ \vec{e}_2 = \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix},\ \dots,\ \vec{e}_n = \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$$

e.g. in $\mathbb{R}^3$, $\vec{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \vec{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \vec{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

Note that the coordinates of any vector $\vec{x} = \begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}$ in the canonical basis are just the entries of $\vec{x}$:

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n$$

E.g. in $\mathbb{R}^3$, $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = x_1\begin{pmatrix}1\\0\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\0\end{pmatrix} + x_3\begin{pmatrix}0\\0\\1\end{pmatrix}$