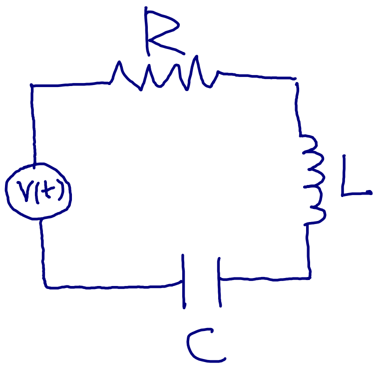
§4.6. Electrical circuits, resonance, and stability

In this section we explore some applications of the theory of inhomogeneous ODEs from §4.5

§4.6.1. RLC circuits

Assume that we have a circuit:



- $V(t)$ external voltage (e.g. battery or receiving antenna)
- R resistor
- L inductor (coil)
- C capacitor

We use the following notation:

- t = time in seconds
- $V(t)$ = external voltage at time t (in volts)
- $0 < R$ = resistance of the resistor (in ohms)
- $0 < L$ = inductance of the inductor (in henries)
- $0 < C$ = capacitance of the capacitor (in farads)
- $I(t)$ = the current at time t (in ampères)

We want to write an ODE for $I(t)$.

To model circuits, we use the following laws:

- Kirchhoff's law: the voltage gain from the power source = sum of voltage drops along the components:

$$V(t) = V_R(t) + V_L(t) + V_C(t)$$

- Voltage drop across the resistor:

$$V_R(t) = R \cdot I(t)$$

- Voltage drop across the inductor:

$$V_L(t) = L \cdot I'(t)$$

- Voltage drop across the capacitor:

$$C \cdot V_C'(t) = I(t)$$

Combining the above, we get

$$V_C(t) = V(t) - R \cdot I(t) - L \cdot I'(t)$$

Differentiating, we get

$$\frac{1}{C} I(t) = V_C'(t) = V'(t) - R \cdot I'(t) - L \cdot I''(t)$$

We arrive to the <u>RLC circuit ODE:</u>

$$\boxed{L \cdot I'' + R \cdot I' + \frac{1}{C} I = V'(t)}$$

This is a 2^{nd} order linear inhomogeneous ODE with constant coefficients

The corresponding homogeneous ODE

$$L \cdot I'' + R \cdot I' + \frac{1}{C} I = 0$$

is the <u>damped harmonic oscillator</u> from §4.2.3.

$(L \to m,\ R \to b,\ \frac{1}{C} \to k)$

In particular,

- if $R = 0$ and $V(t) = 0$ then we get the harmonic oscillator $L \cdot I'' + \frac{1}{C} I = 0$ whose general solution is
$$I = C_1 \cos(\omega t) + C_2 \sin(\omega t), \qquad \omega := \frac{1}{\sqrt{LC}}$$
- if $V(t)$ is exponential $(V(t) = A e^{z_0 t})$ or sinusoidal $(V(t) = A\cos(\omega_0 t - \varphi))$ or a mix of both $(V(t) = A e^{pt} \cos(\omega_0 t - \varphi))$ then we can find the general solution using the methods of §4.5.

§4.6.2. Resonance

Here we look at a damped harmonic oscillator with sinusoidal input. We use the example
$$\boxed{y'' + by' + 9y = \cos(\omega t)} \qquad (b \geq 0,\ \omega > 0)$$
From §4.5.5 we know that a particular solution is given by
$$y(t) = r\cos(\omega t - \theta)$$
where θ is the phase lag (we won't use it here) and r is the <u>amplitude gain</u>:
$$r = \frac{1}{|P(i\omega)|} \quad \text{where } P(z) = z^2 + bz + 9$$

We study the dependence of r on ω. In application to RLC circuits this tells us how much the incoming signal is amplified by the circuit depending on the incoming frequency.

Problem 1: Assume $b=4$. Which value of ω gives the maximal amplitude gain r?

TECHNIQUE PRACTICE

Solution: We compute

$$r = \frac{1}{|P(i\omega)|} = \frac{1}{|-\omega^2+4i\omega+9|} = \frac{1}{|9-\omega^2+4i\omega|}$$

$$= \frac{1}{\sqrt{(9-\omega^2)^2+16\omega^2}}.$$

To maximize r we need to minimize

$$(9-\omega^2)^2+16\omega^2 = \omega^4 - 2\omega^2 + 81$$

Completing the square, we get

$$(\omega^2-1)^2+80.$$

The minimum is achieved when $\omega^2=1$, which means (recalling that $\omega>0$) $\boxed{\omega=1}$

The maximal value of r is

$$\boxed{r = \frac{1}{\sqrt{80}} = \frac{1}{4\sqrt{5}}.}$$

See the course website for graphs of $r(\omega)$ for different values of b

Now assume that $\underline{b=0}$ (no damping)

Then $r = \dfrac{1}{|P(i\omega)|} = \dfrac{1}{|9-\omega^2|}$.

We see that the amplitude gain becomes <u>infinite</u> when $\omega = 3$, i.e. when the input frequency is equal to the frequency of oscillations of the harmonic oscillator itself.

This is known as a <u>resonance phenomenon</u>.

If $\omega = 3$ we cannot use ERF because $P(i\omega)=0$.

So we use generalized ERF:

<u>Problem 2</u> Find the general solution to the ODE

$$y'' + 9y = \cos(3t)$$

<u>Solution</u> We use <u>generalized ERF</u> + <u>complex replacement</u>

$\cos(3t) = \mathrm{Re}\left(e^{3it}\right)$

$P(z) = z^2 + 9$, $z_0 = 3i$, $\boxed{P(z_0) = 0}$

$P'(z) = 2z$, $\boxed{P'(z_0) = 6i}$

A particular solution to $y'' + y = e^{3it}$ is given by $y_c(t) = \dfrac{te^{3it}}{6i} = -\dfrac{i}{6}te^{3it}$.

A particular solution to $y'' + y = \cos(3t)$ is then

$\mathrm{Re}\, y_c(t) = \mathrm{Re}\left(-\frac{i}{6}t(\cos(3t) + i\sin(3t))\right) = \dfrac{t\sin(3t)}{6}$.

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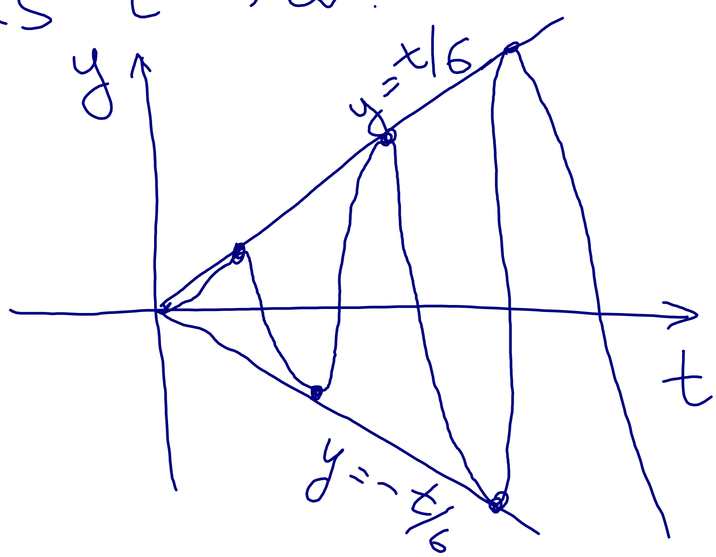
Adding to this the general solution to the homogeneous equation $y''+9y=0$ we get the answer

$$\boxed{\frac{t\sin(3t)}{6}+C_1\cos(3t)+C_2\sin(3t).}$$

Note that $\dfrac{t\sin(3t)}{6}$ grows unbounded as $t\to\infty$:



That is, exciting an undamped harmonic oscillator at resonant frequency leads to unboundedly growing amplitude

§4.6.3. Stability

Consider a general ODE of the form

$P(D)y = b(t)$ where

$P(z) = a_k z^k + \cdots + a_1 z + a_0$

Definition We say that the ODE above is (asymptotically) stable, if every solution $y(t)$ to the homogeneous ODE $P(D)y = 0$ converges to 0 as $t \to \infty$:

$$\lim_{t\to\infty} y(t) = 0.$$

THEORY

If the ODE is stable, then the general solution of $P(D)y(t) = b(t)$ has the form $\tilde{y}(t) + \hat{y}(t)$ where $\tilde{y}(t)$ is a particular solution to $P(D)\tilde{y}(t) = b(t)$ and $\hat{y}(t)$, being a solution to $P(D)\hat{y}(t) = 0$, converges to 0 as $t \to \infty$. If we are solving an initial value problem (IVP) then

$$y(t) = \tilde{y}(t) + \hat{y}(t)$$

$y(t)$: solution to IVP

$\tilde{y}(t)$: steady-state solution independent of the initial condition

$\hat{y}(t)$: transient solution depends on the initial condition but $\to 0$ as $t \to \infty$

Test for stability

TECHNIQUE

The equation $P(D)y = b(t)$ is stable if and only if every characteristic root z (i.e. solution to $P(z)=0$) satisfies $\boxed{\operatorname{Re} z < 0.}$

Proof (sketch) As we learned in §4.4, the general solution to $P(D)y=0$ is a linear combination of functions of the form $t^{\ell}e^{zt}$ where z is a real characteristic root and $t^{\ell}e^{pt}\cos(qt)$, $t^{\ell}e^{pt}\sin(qt)$ where $z = p \pm iq$ are complex characteristic roots.

These functions $\to 0$ as $t\to\infty$ if and only if
$z<0$ (real root) or
$p = \operatorname{Re} z < 0$ (complex roots). $\square$

Test for 2^{nd} order equations:

TECHNIQUE

If $P(z) = a_2 z^2 + a_1 z + a_0$, a_0, a_1, a_2 real, $a_2>0$, then $P(D)y = b(t)$ is stable if and only if

$$\boxed{a_0>0,\ a_1>0}$$

Proof Dividing $P(z)$ by a_2 we reduce to the case $\boxed{a_2=1}$

See MITx 1.10.13 for a different proof

We use Vieta's formulas:
if z_1, z_2 are the roots of
$$z^2 + a_1 z + a_0 = 0 \quad \text{then}$$
$$z_1 + z_2 = -a_1, \qquad z_1 \cdot z_2 = a_0.$$

- If we have 2 complex (non-real) roots $z_{1,2} = p \pm iq$, then $\operatorname{Re} z_{1,2} = p = -\frac{a_1}{2}$

So stability $\Leftrightarrow$ $a_1 > 0$

And $a_0 = (p+iq)(p-iq) = p^2 + q^2 > 0$ always

- If we have 2 real roots (or a double real root) then stability $\Leftrightarrow$ $z_1 < 0$ and $z_2 < 0$.

For that we certainly need $z_1 + z_2 = -a_1 < 0$, i.e. $\boxed{a_1 > 0}$. Now let's assume $a_1 > 0$.

Then at least one of z_1, z_2 is negative (since $z_1 + z_2 = -a_1 < 0$). Let's say $z_1 < 0$.

Then stability $\Leftrightarrow$ $z_2 < 0 \Leftrightarrow z_1 \cdot z_2 > 0 \Leftrightarrow a_0 > 0$. □

<u>Example 1</u> Damped harmonic oscillator

$$m \cdot y'' + b \cdot y' + k \cdot y = b(t), \qquad m, k > 0, \; b \geq 0$$

PRACTICE

<u>$b > 0$:</u> stable

<u>$b = 0$:</u> not stable

Example 2 $y'' = y$, i.e.

$y'' - y = 0$.

Unstable by the Test for 2^{nd} order eqns (as $a_0 = -1 < 0$)

Can also see it from the general solution

$$y(t) = C_1 e^t + C_2 e^{-t} \underset{t\to\infty}{\not\to} 0 \quad \text{if } C_1 \neq 0.$$

Equations like the one in Example 2 model the motion of a point particle near the top of a hill:

Example 3: $y''' + y = 0$.

$P(z) = z^3 + 1 \to$ roots $-1, \ \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ (see §3)

Unstable since $\mathrm{Re}\left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right) = \frac{1}{2} > 0$