§4.5. Inhomogeneous equations

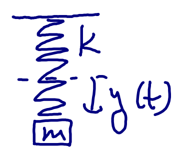
In this section we study linear inhomogeneous constant coefficient ODES

$$\boxed{a_k y^{(k)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = b(t)} \quad \text{(NH)}$$

where $a_0, \ldots, a_k$ are constants, $a_k \neq 0$, and $b(t)$ is a function.

If we are modeling using Newton's IInd Law then $b(t)$ is the external force:

Example: vertical spring



$g \downarrow$ gravity (in $\frac{\text{meter}}{\text{sec}^2}$)

Newton's IInd law:

$$m \cdot y''(t) = F_{\text{spring}} + F_{\text{gravity}} = -k\,y(t) + m \cdot g$$

Get the equation

$$\boxed{my'' + ky = mg}$$

We use below as an example the special case $m=1$, $k=1$, $g=1$:

$$\boxed{y'' + y = 1}$$

§4.5.1. Operator notation

We will use the notation D for the differentiation operator: if y is a function then

$$D\,y = y'.$$

We can multiply D by itself and by constants, e.g.

$$(3D^2)\,y = 3y''.$$

Now the equation (NH) has the form

$$\boxed{P(D)y = b(t)} \qquad \text{(NH)}$$

where P is the characteristic polynomial defined by

$$\boxed{P(z) = a_k z^k + \dots + a_1 z + a_0}$$

We use the corresponding homogeneous ODE

$$\boxed{P(D)y = 0} \qquad \text{(H)}$$

Note: characteristic roots of (H) are solutions to $P(z) = 0$.

Example: $y'' + y = 1$ (NH) $\qquad P(z) = z^2 + 1$

$y'' + y = 0$ (H)

THEORY

§4.5.2. Superposition Principle

Remember that in §4.1.3 THEORY we had a useful formula for the general solution of the ODE (H).

Now we give a formula for the general solution to (NH):

Theorem 1. Assume that y_1, y_2 solve

$$\boxed{P(D)\,y_1 = b_1(t)} \quad (NH_1)$$

$$\boxed{P(D)\,y_2 = b_2(t)} \quad (NH_2)$$

Then for any constants C_1, C_2, the function $C_1y_1 + C_2y_2$ solves

$$\boxed{P(D)\,(C_1y_1 + C_2y_2) = C_1b_1(t) + C_2b_2(t)}$$

(same is true for a linear combination of more than 2 fns)

2. Assume that $\tilde{y}(t)$ is some solution to

$$\boxed{P(D)\,\tilde{y}(t) = b(t)} \quad (NH)$$

Then the general solution to (NH) has the form

$$y(t) = \tilde{y}(t) + (\text{general solution to (H)})$$

where (H) is the homogeneous equation

$$P(D)\,y = 0 \quad (H)$$

Proof 1. We compute

$$P(D)(C_1y_1+C_2y_2)$$
$$= C_1P(D)y_1 + C_2P(D)y_2$$
$$= C_1b_1(t) + C_2b_2(t)$$

a calculation needed here...

2. If $\tilde{y}$ and y both solve (NH) then by part 1 (or by direct calculation) the difference $y - \tilde{y}$ solves (H). So each solution to (NH) has the form

$$\tilde{y} + (\text{solution to (H)}) \qquad (*)$$

Also, if $\hat{y}$ solves (H) then by part 1, the function $\tilde{y}+\hat{y}$ solves (NH). So the formula $(*)$ gives all solutions to (NH). $\square$

Remark: as before in §4.1, this proof (presented here very briefly) relies on the fact that (NH) is a linear equation.

The theorem above means that

TECHNIQUE

(1) in order to find the general solution of (NH) it suffices to find one solution of (NH) (often called a particular solution) and the general solution of (H) and add them up

(2) if the forcing term $b(t)$ on the right-hand side of (NH) is the sum of several functions then we can solve (NH) with each of these functions separately and add the results

Example $y'' + y = 1 + e^t$ (NH) PRACTICE

Imagine that we guessed some solutions (a systematic procedure will come later):

$y_1 = 1$ solves $y_1'' + y_1 = 1$

$y_2 = \frac{1}{2}e^t$ solves $y_2'' + y_2 = e^t$

Then $\tilde{y} = y_1 + y_2 = 1 + \frac{e^t}{2}$ solves $\tilde{y}'' + \tilde{y} = 1 + e^t$ (NH)

Homogeneous equation: $y'' + y = 0$ (H)

General solution to (H) is $C_1 \cos t + C_2 \sin t$

General solution to (NH) is $\boxed{1 + \frac{e^t}{2} + C_1 \cos t + C_2 \sin t}$

§4.5.3. Exponential Response Formula (ERF)

We won't learn how to solve (NH) with an arbitrary right-hand side $b(t)$ (this is possible with variation of parameters but quite tedious...)

Instead we consider some useful and easier to handle right-hand sides $b(t)$.

We start with $b(t)$ which is exponential:

$$\boxed{b(t) = e^{z_0 t}}$$

where z_0 is a (possibly complex) constant

THEORY

So we are solving

$$\boxed{P(D)y = e^{z_0 t}}$$

where recall that

$$P(D)y = a_k y^{(k)} + \cdots + a_1 y' + a_0 y.$$

We look for a particular solution in the form

$\boxed{y = c e^{z_0 t}}$ where c is a constant.

Substituting into the equation we get

$P(D)y = c \cdot P(z_0) e^{z_0 t}$ where

$P(z) = a_k z^k + \cdots + a_1 z + a_0$ is the characteristic polynomial

So we need $c \cdot P(z_0) e^{z_0 t} = P(D)y = e^{z_0 t}$

If $P(z_0) \neq 0$ we can solve for c and get

$c = 1/P(z_0)$. This gives

Exponential Response Formula:

Assume $P(D)$, $P(z)$ are as above and z_0 is a complex constant such that $P(z_0) \neq 0$.

Then the function

$$\boxed{y(t) = \frac{1}{P(z_0)} e^{z_0 \cdot t}}$$

solves the ODE

$$\boxed{P(D)\, y = e^{z_0 \cdot t}}$$

Example 1: $y'' + y = 1 = e^{0 \cdot t}$ → $z_0 = 0$

$P(z) = z^2 + 1$, $P(z_0) = 1 \neq 0$

The function $\boxed{y = \frac{1}{1} e^{0 \cdot t} = 1}$ is a (particular) solution to the ODE.

The general solution is $y = 1 + C_1 \cos t + C_2 \sin t$

Example 2: $y'' + y = e^t = e^{1 \cdot t}$ → $z_0 = 1$

$P(z) = z^2 + 1$, $P(z_0) = 2 \neq 0$

The function $y = \frac{1}{2} \cdot e^{1 \cdot t} = \frac{e^t}{2}$ is a (particular) solution to the ODE.

The general solution is $y = \frac{e^t}{2} + C_1 \cos t + C_2 \sin t$

§4.5.4. Complex replacement

Now we consider the right-hand side $b(t)$ which is a sinusoidal function

$$\boxed{b(t) = B_1 \cos(\omega t) + B_2 \sin(\omega t)}$$

where $\omega > 0$, B_1, B_2 are real constants

We use ERF + complex replacement:

ALGORITHM for finding a particular solution to $P(D)y = b(t)$ where $b(t)$ is sinusoidal:

TECHNIQUE

Step 1: Write $b(t)$ as the real part of a complex exponential

$b(t) = \mathrm{Re}(ce^{i\omega t})$ where $\boxed{c = B_1 - iB_2}$

Step 2: use ERF for the complex exponential with $z_0 = i\omega$

$y_c(t) = \dfrac{ce^{i\omega t}}{P(i\omega)}$ is a solution to $P(D)y_c = ce^{i\omega t}$

Step 3: take the real part of y_c:

$y(t) = \mathrm{Re}\left(\dfrac{ce^{i\omega t}}{P(i\omega)}\right)$ is a solution to

$P(D)y(t) = b(t)$.

PRACTICE

Example:

$$y'' + y = -\cos(2t) + \sqrt{3}\sin(2t)$$

1. Write in the complex form: $B_1 = -1,\ B_2 = \sqrt{3} \to c = -1-\sqrt{3}i,$

$$-\cos(2t) + \sqrt{3}\sin(2t) = \mathrm{Re}\left((-1-\sqrt{3}i)e^{2it}\right)$$

(Check this part: $\mathrm{Re}\left((-1-\sqrt{3}i)e^{2it}\right)$

$= \mathrm{Re}\left((-1-\sqrt{3}i)(\cos(2t) + i\sin(2t))\right)$

$= -\cos(2t) + \sqrt{3}\sin(2t)$ indeed)

2. Use ERF: $P(z) = z^2 + 1,\ z_0 = 2i,$

$$P(z_0) = (2i)^2 + 1 = -4 + 1 = -3$$

Thus $y_c(t) = \dfrac{(-1-\sqrt{3}i)e^{2it}}{-3} = \dfrac{(1+\sqrt{3}i)e^{2it}}{3}$

Solves $y_c'' + y_c = (-1-\sqrt{3}i)e^{2it}$.

3. Take the real part: a particular solution to the ODE $y'' + y = -\cos(2t) + \sqrt{3}\sin(2t)$ is given by

$$y(t) = \mathrm{Re}\, y_c(t) = \mathrm{Re}\left(\frac{1+\sqrt{3}i}{3} \cdot (\cos(2t) + i\sin(2t))\right)$$

$$= \frac{\cos(2t) - \sqrt{3}\sin(2t)}{3}.$$

The general solution is

$$\frac{\cos(2t) - \sqrt{3}\sin(2t)}{3} + C_1\cos t + C_2\sin t.$$

Note: same technique works for

$$b(t) = e^{pt} \cdot (B_1\cos(\omega t) + B_2\sin(\omega t)) = \mathrm{Re}\left((B_1 - iB_2)e^{(p+i\omega)t}\right)$$

Apply ERF with $z_0 = p + i\omega$

See below for an example

§4.5.5. Complex gain & phase lag

We can also do complex replacement when $b(t)$ is in the amplitude-phase form:

$$b(t) = A\cos(\omega t - \varphi)$$

$$b(t) = \mathrm{Re}\left(Ae^{i(\omega t-\varphi)}\right) = \mathrm{Re}\left(Ae^{-i\varphi}e^{i\omega t}\right)$$

ERF $\Rightarrow$ a solution to $P(D) = Ae^{-i\varphi}e^{i\omega t}$ is given by $y_c(t) = \dfrac{Ae^{-i\varphi}}{P(i\omega)}e^{i\omega t}$

A solution to $P(D)y = A\cos(\omega t-\varphi)$ is then given by

TECHNIQUE

$$\boxed{y(t) = \mathrm{Re}\, y_c(t) = \mathrm{Re}\left(\frac{Ae^{-i\varphi}}{P(i\omega)}e^{i\omega t}\right).}$$

How to write this back in the amplitude-phase form?

We write $\dfrac{1}{P(i\omega)}$ in the polar form:

$$\boxed{\frac{1}{P(i\omega)} = re^{i\theta}.}$$

Then

$$y(t) = \mathrm{Re}\left(A\cdot re^{i(\theta-\varphi)}e^{i\omega t}\right) = \mathrm{Re}\left(A\cdot r\, e^{i(\theta-\varphi+\omega t)}\right)$$

$$= Ar\cos(\omega t - (\varphi-\theta)).$$

We now compare the amplitude-phase representations of the forcing term $b(t)$ and the resulting solution $y(t)$:

Forcing term $b(t)$ (input signal)	Solution (output signal) $y(t)$	
ω	ω	Frequency
A	$A \cdot r$	Amplitude
φ	$\varphi - \theta$	Phase

So the amplitude is multiplied by r and the phase is shifted by $-\theta$.

We define:

- Complex gain $= \dfrac{1}{P(i\omega)}$
- Amplitude gain $= r$
- Phase lag $= -\theta$

where $\boxed{\dfrac{1}{P(i\omega)} = re^{i\theta}}$

Same works for damped sinusoidal inputs

$$Ae^{pt}\cos(\omega t - \varphi) = \mathrm{Re}\left(Ae^{-i\varphi}e^{(p+i\omega)t}\right)$$

with $P(i\omega)$ replaced by $P(p+i\omega)$.

Example: Find a particular solution, complex gain, amplitude gain, and phase lag for the ODE

$$y'' - y = 3e^{t}\cos(2t)$$

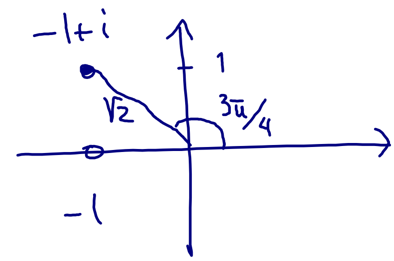
Solution: write in the complex form

$$3e^t\cos(2t) = \mathrm{Re}\left(3e^{(1+2i)t}\right).$$

Use ERF: $P(z) = z^2-1$, $z_0 = 1+2i$,

$$P(z_0) = (1+2i)^2 - 1 = -4+4i$$

Thus $\boxed{y_c(t) = \dfrac{3}{-4+4i}\, e^{(1+2i)t}}$



solves $y_c'' - y_c = 3e^{(1+2i)t}$.

Write $\dfrac{1}{P(z_0)} = \dfrac{1}{-4+4i}$ in polar form:

$-4+4i = 4(-1+i)$ and $|-1+i| = \sqrt{2}$, $\arg(-1+i) = 3\pi/4$

So $-1+i = \sqrt{2}\, e^{\frac{3\pi i}{4}}$, thus

$$\frac{1}{-4+4i} = \frac{1}{4\sqrt{2}}\, e^{-\frac{3\pi i}{4}}.$$

Now $\boxed{y_c(t) = \dfrac{3}{4\sqrt{2}}\, e^{-\frac{3\pi i}{4}}\, e^{(1+2i)t}}$

and a particular solution to $y'' - y = 3e^t\cos(2t)$ is given by $y(t) = \mathrm{Re}(y_c(t)) = \dfrac{3}{4\sqrt{2}}\mathrm{Re}\left(e^t e^{i(2t-\frac{3\pi}{4})}\right)$

$$= \boxed{\frac{3}{4\sqrt{2}}\, e^t \cos\left(2t - \frac{3\pi}{4}\right)}$$

Complex gain $= 1/P(z_0) = \dfrac{1}{-4+4i}$

Amplitude gain $= \left|\dfrac{1}{P(z_0)}\right| = \dfrac{1}{4\sqrt{2}}$

Phase lag $= -\arg(1/P(z_0)) = \dfrac{3\pi}{4}$.

§4.5.6. Generalized ERF

Recall ERF: if $P(z_0) \neq 0$

then $y(t) = \dfrac{e^{z_0 t}}{P(z_0)}$ solves the ODE

THEORY

$$\boxed{P(D)y(t) = e^{z_0 t}} \quad (NH)$$

What if $P(z_0) = 0$? Then no multiple of $e^{z_0 t}$ solves (NH) because $P(D)e^{z_0 t} = 0$

($e^{z_0 t}$ solves the homogeneous equation)

Instead we multiply by t and use

Theorem (Generalized ERF)

Assume that $P(z_0) = 0$ and $P'(z_0) \neq 0$.

Then the function $\boxed{y(t) = \dfrac{t e^{z_0 \cdot t}}{P'(z_0)}}$

satisfies the ODE

$$\boxed{P(D)y = e^{z_0 \cdot t}.}$$

The proof below is totally optional...

Proof We factorize: $P(z) = Q(z)\cdot(z - z_0)$.

This is possible since z_0 is a root of P.

Now $P(D)y = Q(D)(D - z_0)y = Q(D)(y' - z_0 y)$

Substitute $y = te^{z_0 t}$, we get $y' - z_0 y = e^{z_0 t}$, so

$P(D)y = Q(D)(y' - z_0 y) = Q(D)e^{z_0 t} = Q(z_0)e^{z_0 t}$.

But $P'(z) = (z - z_0)Q'(z) + Q(z) \Rightarrow P'(z_0) = Q(z_0)$. $\square$

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Note: MITx 1.9.10 has another proof using partial derivatives.

Generalized ERF works with complex replacement (example in §4.6)

Example: $y''-y=3e^t = 3e^{1\cdot t}$ (NH)

$P(z)=z^2-1$, $z_0=1$, $\boxed{P(z_0)=0}$

$P'(z)=2z$, $P'(z_0)=2$

Use generalized ERF: get a particular solution

$$y(t)=\frac{3}{2}te^t.$$

General solution to $y''-y=0$ is $C_1e^t+C_2e^{-t}$.

Thus the general solution to (NH) is

$$y(t)=\frac{3}{2}te^t+C_1e^t+C_2e^{-t}.$$

What if $P(z_0)=0$ and $P'(z_0)=0$?

Theorem (Even more generalized ERF) Assume that $P(z_0)=P'(z_0)=\dots=P^{(m-1)}(z_0)=0$, $P^{(m)}(z_0)\neq 0$.

Then a solution to $P(D)y=e^{z_0t}$ is given by $y=\dfrac{t^me^{z_0t}}{P^{(m)}(z_0)}$

Proof: see MITx 1.9.10