§4.5. Inhomogeneous equations

In this section we study linear inhomogeneous constant coefficient ODEs
\[ a_k y^{(k)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = b(t) \quad (NH) \]
where \( a_0, \ldots, a_k \) are constants, \( a_k \neq 0 \), and \( b(t) \) is a function.

If we are modeling using Newton's IInd Law then \( b(t) \) is the **external force**:

**Example**: vertical spring
\[ \text{Newton's IInd law:} \quad m \cdot y''(t) = F_{\text{spring}} + F_{\text{gravity}} = -k y(t) + mg \]
Get the equation
\[ m y'' + ky = mg \]
We use below as an example the special case \( m=1 \), \( k=1 \), \( g=1 \):
\[ y'' + y = 1 \]
§4.5.1. Operator notation

We will use the notation $D$ for the differentiation operator: if $y$ is a function then

$$D\ y = y'.$$

We can multiply $D$ by itself and by constants, e.g.

$$(3D^2)\ y = 3y''.$$  

Now the equation \((NH)\) has the form

$$P(D)y = b(t),$$

where $P$ is the characteristic polynomial defined by

$$P(t) = a_k t^k + \ldots + a_1 t + a_0.$$  

We use the corresponding homogeneous ODE

$$P(D)y = 0$$

Note: characteristic roots of \((H)\) are solutions to $P(t) = 0$.

Example: \(y'' + y = 1\ \ (NH)\)  
\(y'' + y = 0\ \ (H)\)  
\[P(t) = t^2 + 1\]
Remember that in §4.1.3 we had a useful formula for the general solution of the ODE (H).
Now we give a formula for the general solution to (NH):

**Theorem 1.** Assume that \( y_1, y_2 \) solve

\[
P(D) y_1 = b_1(t) \quad \text{(NH1)}
\]
\[
P(D) y_2 = b_2(t) \quad \text{(NH2)}
\]

Then for any constants \( C_1, C_2 \), the function \( C_1 y_1 + C_2 y_2 \) solves

\[
P(D) (C_1 y_1 + C_2 y_2) = C_1 b_1(t) + C_2 b_2(t)
\]
(same is true for a linear combination of more than 2 fns)

2. Assume that \( \tilde{y}(t) \) is some solution to

\[
P(D) \tilde{y}(t) = b(t) \quad \text{(NH)}
\]

Then the general solution to (NH) has the form

\[y(t) = \tilde{y}(t) + \text{(general solution to (H))} \]

where (H) is the homogeneous equation

\[
P(D) y = 0 \quad \text{(H)}
\]
Proof 1. We compute

\[ P(D)(C_1 y_1 + C_2 y_2) \]
\[ = C_1 P(D)y_1 + C_2 P(D)y_2 \]

a calculation needed here...

2. If \( \tilde{y} \) and \( y \) both solve \((NH)\)
then by part 1 (or by direct calculation)
the difference \( y - \tilde{y} \) solves \((H)\).
So each solution to \((NH)\) has the form
\( \tilde{y} + (\text{solution to } (H)) \)
\((*)\)
Also, if \( \hat{y} \) solves \((H)\) then by part 1,
the function \( \tilde{y} + \hat{y} \) solves \((NH)\).
So the formula \((*)\) gives all solutions to \((NH)\). \(\square\)

Remark: as before in §4.1, this proof
(presented here very briefly) relies
on the fact that \((NH)\) is a
linear equation.
The theorem above means that (1) in order to find the general solution of (NH) it suffices to find one solution of (NH) (often called a particular solution) and the general solution of (H) and add them up.

(2) If the forcing term $b(t)$ on the right-hand side of (NH) is the sum of several functions, then we can solve (NH) with each of these functions separately and add the results.

Example $y'' + y = 1 + e^t$ (NH)

Imagine that we guessed some solutions (a systematic procedure will come later):

$y_1 = 1$ solves $y''_1 + y_1 = 1$

$y_2 = \frac{1}{2}e^t$ solves $y''_2 + y_2 = e^t$

Then $\tilde{y} = y_1 + y_2 = 1 + \frac{1}{2}e^t$ solves $\tilde{y}'' + \tilde{y} = 1 + e^t$ (NH)

Homogeneous equation: $y'' + y = 0$ (H)

General solution to (H) is $C_1 \cos t + C_2 \sin t$

General solution to (NH) is $\frac{1 + \frac{1}{2}e^t}{2} + C_1 \cos t + C_2 \sin t$
We won't learn how to solve (NH) with an arbitrary right-hand side \( b(t) \) (this is possible with variation of parameters but quite tedious...).

Instead we consider some useful and easier to handle right-hand sides \( b(t) \).

We start with \( b(t) \) which is exponential:

\[
b(t) = e^{z_0 \cdot t}
\]

where \( z_0 \) is a (possibly complex) constant.

So we are solving

\[
P(D)y = e^{z_0 \cdot t}
\]

where recall that

\[
P(D)y = a_k y^{(k)} + \ldots + a_1 y' + a_0 y.
\]

We look for a particular solution in the form

\[
y = c e^{z_0 t}
\]

where \( c \) is a constant.

Substituting into the equation we get

\[
P(D)y = c \cdot P(z_0) e^{z_0 t}
\]

where

\[
P(z) = a_k z^k + \ldots + a_1 z + a_0
\]

is the characteristic polynomial.

So we need

\[
c \cdot P(z_0) e^{z_0 t} = P(D)y = e^{z_0 t}
\]

If \( P(z_0) \neq 0 \) we can solve for \( c \) and get

\[
c = \frac{1}{P(z_0)}.
\]

This gives
Exponential Response Formula:

Assume $P(z)$, $P(z)$ are as above and $z_0$ is a complex constant such that $P(z_0) 
eq 0$. Then the function

$$y(t) = \frac{1}{P(z_0)} e^{z_0 t}$$

solves the ODE

$$P(z) y = e^{z_0 t}$$

Example 1: $y'' + y = 1 = e^{0 t} \quad \Rightarrow (z_0 = 0)$

$P(z) = z^2 + 1$, $P(z_0) = 1 \neq 0$

The function $y = \frac{1}{1} e^{0 t} = 1$ is a (particular) solution to the ODE.

The general solution is $y = 1 + C_1 \cos t + C_2 \sin t$

Example 2: $y'' + y = e^t = e^{1 t} \quad \Rightarrow (z_0 = 1)$

$P(z) = z^2 + 1$, $P(z_0) = 2 \neq 0$

The function $y = \frac{1}{2} e^{1 t} = \frac{e^t}{2}$ is a (particular) solution to the ODE.

The general solution is $y = \frac{e^t}{2} + C_1 \cos t + C_2 \sin t$
§4.5.4. Complex replacement

Now we consider the right-hand side \( b(t) \) which is a sinusoidal function

\[
 b(t) = B_1 \cos(\omega t) + B_2 \sin(\omega t)
\]

where \( \omega > 0, B_1, B_2 \) are real constants. We use ERF + complex replacement.

**Algorithm** for finding a particular solution to \( P(D)y = b(t) \) where \( b(t) \) is sinusoidal:

**Technique**

**Step 1:** Write \( b(t) \) as the real part of a complex exponential

\[
 b(t) = \text{Re} \left( ce^{i\omega t} \right)
\]

where \( c = B_1 - iB_2 \)

**Step 2:** Use ERF for the complex exponential with \( z_0 = i\omega \)

\[
 y_c(t) = \frac{ce^{i\omega t}}{P(i\omega)}
\]

is a solution to \( P(D)y_c = ce^{i\omega t} \)

**Step 3:** Take the real part of \( y_c \):

\[
 y(t) = \text{Re} \left( \frac{ce^{i\omega t}}{P(i\omega)} \right)
\]

is a solution to \( P(D)y(t) = b(t) \).
Example:

\[ y'' + y = -\cos(2t) + \sqrt{3}\sin(2t) \]

1. Write in the complex form: \( B_1 = -1, B_2 = \sqrt{3} \Rightarrow c = -(1 - \sqrt{3}i) \)

\[ -\cos(2t) + \sqrt{3}\sin(2t) = \text{Re} \left( (1 - 1\sqrt{3}i)e^{2it} \right) \]

(Check this part: \( \text{Re} \left( (1 - 1\sqrt{3}i)e^{2it} \right) \)

\[ = \text{Re} \left( (1 - 1\sqrt{3}i)(\cos(2t) + i\sin(2t)) \right) \]

\[ = -\cos(2t) + \sqrt{3}\sin(2t) \text{ indeed} \]

2. Use ERF: \( P(z) = z^2 + 1, \quad z_0 = 2i \)

\[ P(z_0) = (2i)^2 + 1 = -4 + 1 = -3 \]

Thus \( y_c(t) = \frac{(-1 - \sqrt{3}i)e^{2it}}{-3} = \frac{(1 + \sqrt{3}i)e^{2it}}{3} \)

Solves \( y'' + y_c = (-1 + \sqrt{3}i)e^{2it} \).

3. Take the real part: a particular solution to the ODE \( y'' + y = -\cos(2t) + \sqrt{3}\sin(2t) \) is given by

\[ y(t) = \text{Re} y_c(t) = \text{Re} \left( \frac{1 + \sqrt{3}i}{3} \right) \left( \cos(2t) + i\sin(2t) \right) \]

\[ = \frac{\cos(2t) - \sqrt{3}\sin(2t)}{3} \]

The general solution is

\[ \frac{\cos(2t) - \sqrt{3}\sin(2t)}{3} + C_1\cos t + C_2\sin t \]

Note: Same technique works for

\[ b(t) = e^{pt} \left( B_1\cos(wt) + B_2\sin(wt) \right) = \text{Re} \left( (B_1 - iB_2)e^{(p+iw)t} \right) \]

Apply ERF with \( z_0 = p + iw \)

See below for an example
§4.5.5. Complex gain & phase lag

We can also do complex replacement when \( b(t) \) is in the amplitude–phase form:

\[
b(t) = A \cos(\omega t - \phi) \]
\[
b(t) = \text{Re} \left( A e^{i(\omega t - \phi)} \right) = \text{Re} \left( A e^{-i\phi} e^{i\omega t} \right) \]

\[\text{ERF} \Rightarrow \text{a solution to } P(D) = A e^{-i\phi} e^{i\omega t} \]
is given by

\[
y_{c}(t) = \frac{A e^{-i\phi}}{P(i\omega)} e^{i\omega t} \]

A solution to \( P(D) y = A \cos(\omega t - \phi) \)
is then given by

\[
y(t) = \text{Re}y_{c}(t) = \text{Re} \left( \frac{A e^{-i\phi}}{P(i\omega)} e^{i\omega t} \right). \]

**TECHNIQUE**

How to write this back in the amplitude–phase form?

We write \( \frac{1}{P(i\omega)} \) in the polar form:

\[
\frac{1}{P(i\omega)} = r e^{i\Theta}. \]

Then

\[
y(t) = \text{Re} \left( A \cdot r e^{i(\Theta - \phi)} e^{i\omega t} \right) = \text{Re} \left( A \cdot r e^{i(\phi - \Theta + \omega t)} \right)
\]

\[= Ar \cos(\omega t - (\phi - \Theta)). \]

We now compare the amplitude–phase representations of the forcing term \( b(t) \) and the resulting solution \( y(t) \):
<table>
<thead>
<tr>
<th>Forcing term $b(t)$ (input signal)</th>
<th>Solution (output signal) $y(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$</td>
<td>$\omega$</td>
</tr>
<tr>
<td>$A$</td>
<td>$A \cdot r$</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>$\varphi - \theta$</td>
</tr>
</tbody>
</table>

So the amplitude is multiplied by $r$ and the phase is shifted by $-\theta$.

We define:

- Complex gain: $\frac{1}{P(i\omega)}$
- Amplitude gain: $r$
- Phase lag: $-\theta$

Same works for damped sinusoidal inputs $A e^{pt} \cos(\omega t - \varphi)$: $\text{Re}(A e^{-i\varphi} e^{(p+i\omega)t})$

with $P(i\omega)$ replaced by $P(p+i\omega)$.

**Example**: Find a particular solution, complex gain, amplitude gain, and phase lag for the ODE

$$y'' - y = 3e^t \cos (2t)$$
Solution: write in the complex form
\[ 3e^t \cos(2t) = \text{Re} \left( 3e^{(1+2i)t} \right). \]

Use ERF: \( P(z) = z^2 - 1, \ z_o = 1+2i; \)
\[ P(z_o) = (1+2i)^2 - 1 = -4 + 4i \]
Thus \[ y_c(t) = \frac{3}{-4+4i} e^{(1+2i)t} \]
solves \( y'' - y_c = 3e^{(1+2i)t} \).

Write \( \frac{1}{P(z_o)} = \frac{1}{-4+4i} \) in polar form:
\[-4 + 4i = 4(-1+i) \text{ and } |-1+i| = \sqrt{2} \]
\[ \text{arg}(-1+i) = \frac{3\pi}{4} \]
So \(-1+i = \sqrt{2} e^{\frac{3\pi i}{4}} \), thus
\[ \frac{1}{-4+4i} = \frac{1}{4\sqrt{2}} e^{-\frac{3\pi i}{4}} \]

Now \[ y_c(t) = \frac{3}{4\sqrt{2}} e^{-\frac{3\pi i}{4}} e^{(1+2i)t} \]
and a particular solution to \( y'' - y = 3e^t \cos(2t) \)
is given by \( y(t) = \text{Re}(y_c(t)) = \frac{3}{4\sqrt{2}} \text{Re}(e^{t} e^{i(2t-\frac{3\pi}{4})}) \)
\[ = \frac{3}{4\sqrt{2}} e^t \cos\left(2t - \frac{3\pi}{4}\right) \]

Complex gain = \( \frac{1}{P(z_o)} = -4 + 4i \)
Amplitude gain = \( \left| \frac{1}{P(z_o)} \right| = \frac{1}{4\sqrt{2}} \)
Phase lag = \( -\text{arg}(\frac{1}{P(z_o)}) = \frac{3\pi}{4} \)
§4.5.6. Generalized ERF

Recall ERF: if \( P(z_0) \neq 0 \)
then \( y(t) = \frac{e^{z_0 t}}{P(z_0)} \) solves the ODE
\[
P(D)y(t) = e^{z_0 t} \quad \text{(NH)}
\]

What if \( P(z_0) = 0 \)? Then no multiple of \( e^{z_0 t} \) solves (NH) because \( P(D)e^{z_0 t} = 0 \) (\( e^{z_0 t} \) solves the homogeneous equation).

Instead we multiply by \( t \) and use

Theorem (Generalized ERF)

Assume that \( P(z_0) = 0 \) and \( P'(z_0) \neq 0 \).
Then the function \( y(t) = \frac{te^{z_0 t}}{P'(z_0)} \)
satisfies the ODE
\[
P(D)y = e^{z_0 t}.
\]

Proof We factorize: \( P(z) = Q(z) \cdot (z - z_0) \).
This is possible since \( z_0 \) is a root of \( P \).

Now \( P(D)y = Q(D)(D - z_0)y = Q(D)(y' - z_0 y) \)
Substitute \( y = te^{z_0 t} \), we get \( y' - z_0 y = e^{z_0 t} \), so
\[
P(D)y = Q(D)(y' - z_0 y) = Q(D)e^{z_0 t} = Q(z_0)e^{z_0 t}.
\]
But \( P'(z) = (z - z_0)Q'(z) + Q(z) \Rightarrow P'(z_0) = Q(z_0) \).

\[ \square \]
Note: MITx 1.9.10 has another proof using partial derivatives.

Generalized ERF works with complex replacement (example in §4.6)

Example: \( y'' - y = 3e^t = 3e^{1+t} \) (NH)

\[ P(z) = z^2 - 1, \quad z_0 = 1, \quad \boxed{P(z_0) = 0} \]

\[ P'(z) = 2z, \quad P'(z_0) = 2 \]

Use generalized ERF: get a particular solution

\[ y(t) = \frac{3}{2} te^t. \]

General solution to \( y'' - y = 0 \) is \( C_1e^t + C_2e^{-t} \).

Thus the general solution to \( (NH) \) is

\[ y(t) = \frac{3}{2} te^t + C_1e^t + C_2e^{-t}. \]

What if \( P(z_0) = 0 \) and \( P'(z_0) = 0 \)?

Theorem (Even more generalized ERF) Assume that

\[ P(z_0) = P'(z_0) = \ldots = P^{(m-1)}(z_0) = 0, \quad P^{(m)}(z_0) \neq 0. \]

Then a solution to \( P(D)y = e^{z_0 t} \)

is given by \[ y = \frac{t^me^{z_0 t}}{P^{(m)}(z_0)} \]

Proof: see MITx 1.9.10