§4.4. Higher order equations

In this section we briefly explain how to generalize the algorithm of §4.2.2 to a general constant coefficient ODE

$$a_k y^{(k)} + \cdots + a_1 y' + a_0 y = 0 \qquad (*)$$

of any order k. Here $a_0, \ldots, a_k$ are real constants and $a_k \neq 0$.

We will use the following <u>example</u>:

$$y^{(6)} + y^{(4)} - y'' - y = 0$$

TECHNIQUE

<u>ALGORITHM for finding the general solution:</u>

<u>Step 1</u> Write the characteristic equation

$$a_k z^k + \cdots + a_1 z + a_0 = 0$$

I.e. replace j-th derivative by z^j

<u>Example:</u> $y^{(6)} + y^{(4)} - y'' - y = 0 \rightarrow z^6 + z^4 - z^2 - 1 = 0$

<u>Step 2</u>: find the characteristic roots

<u>Example:</u> $z^6 + z^4 - z^2 - 1 = (z^4 - 1)(z^2 + 1)$

$= (z^2 - 1)(z^2 + 1)^2 = (z-1)(z+1)(z-i)^2(z+i)^2$

<u>Roots</u>: $1, -1, i, -i, i, -i$ (with multiplicity)

Step 3: find k linearly independent solutions $y_1, \dots, y_k$ to $(*)$, possibly complex valued.

- For each characteristic root z use the function $e^{z \cdot t}$
- If a root z has multiplicity $\ell \geq 2$, use the functions $e^{z \cdot t}, te^{z \cdot t}, \dots, t^{\ell-1} e^{z \cdot t}$ (keep multiplying by t until we reach the multiplicity)

Example: roots $1, -1, i, -i, i, -i$

$1 \longrightarrow e^{t}$

$-1 \longrightarrow e^{-t}$

$i \longrightarrow e^{it}, te^{it}$ (multiplicity 2)

$-i \longrightarrow e^{-it}, te^{-it}$ (multiplicity 2)

Step 4: if there are complex roots, do a complex switch: ~~for~~ each complex conjugate pair of roots $p+iq, p-iq$, replace $e^{(p+iq)t}, e^{(p-iq)t}$ by

$e^{pt}\cos(qt), e^{pt}\sin(qt)$

Example:

$$\left.\begin{matrix} e^{it} \\ e^{-it} \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \cos t \\ \sin t \end{matrix}\right.$$

$$\left.\begin{matrix} te^{it} \\ te^{-it} \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} t\cos t \\ t\sin t \end{matrix}\right.$$

End up with the functions

$$e^{t}, e^{-t}, \cos t, \sin t, t\cos t, t\sin t$$

Step 5: the general solution is a linear combination

$$c_1 y_1 + \cdots + c_k y_k$$

where $c_1, \ldots, c_k$ are arbitrary constants.

Example: the general solution to the ODE

$$y^{(6)} + y^{(4)} - y'' - y = 0$$

is given by

$$\boxed{y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t + c_5 t\cos t + c_6 t\sin t}$$

(scroll down for more...)

THEORY

18.03 §4.4 4

Existence/uniqueness Theorem

Consider the linear ODE

$$(\star)\; a_k(t)y^{(k)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t)=b(t)$$

where $a_0,\ldots,a_k, b$ are continuous functions on the real line and $a_k(t)\neq 0$ for all t.

Then for any real numbers $A_0,\ldots,A_{k-1}, t_0$ the initial value problem

$$\begin{cases} (\star) \\ y(t_0)=A_0 \\ y'(t_0)=A_1 \\ \vdots \\ y^{(k-1)}(t_0)=A_{k-1} \end{cases}$$

has a unique solution y which is a k times differentiable function

Remark 1. We need k pieces of data for k-th order equation, namely the value of y and its first $k-1$ derivatives at t_0. So for example for a 2nd order equation we need to specify $y(t_0), y'(t_0)$. Solutions to $(\star)$ are parametrized by $y(t_0), y'(t_0), \ldots, y^{(k-1)}(t_0)$.

Remark 2. The existence/uniqueness theorem applies to nonlinear equations as well. But in general the solution $y(t)$ will only be defined on some interval $(t_0 - a, t_0 + b)$ where $a, b > 0$ depend on the initial data $A_0, A_1, \ldots, A_{k-1}$.

Example: $\begin{cases} y' = y^2 \\ y(0) = A \end{cases}$

The solution is $y(t) = \dfrac{1}{C - t}$ where $C = \dfrac{1}{A}$, that is $y(t) = \dfrac{A}{1 - At}$ (this includes the special case $y = 0$)

If $A > 0$ then the solution $y(t)$ is defined on the interval $(-\infty, \frac{1}{A})$.
At $t = 1/A$ the solution blows up.
(The fact that the formula $\dfrac{A}{1-At}$ also makes sense for $t > 1/A$ is irrelevant: we do not continue the solution past the blow up point.)