§4.3. Sinusoidal functions

We encountered in §4.2
functions of the form
$$e^{pt}(C_1\cos(\omega t)+C_2\sin(\omega t))$$
where p, ω, C_1, C_2 are real numbers, $\omega>0$

THEORY

A special case with $p=0$ is called
a sinusoidal function:

Definition A sinusoidal function
is a function of the form
$$\boxed{y(t)=C_1\cos(\omega t)+C_2\sin(\omega t)}$$
where C_1, C_2, ω are real constants and $\omega>0$.

We call ω the frequency of this function

The period is defined to be $\boxed{T=\frac{2\pi}{\omega}}$

Note that y is T-periodic:
$$y(t+T)=y(t).$$

§4.3.1. The three forms of sinusoidal functions

A sinusoidal function can be written in the following 3 equivalent forms:

THEORY

Linear combination: $\boxed{y(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)}$

C_1, C_2 real numbers

Complex form: $\boxed{y(t) = \operatorname{Re}(c e^{i\omega t})}$

c complex number, Re = real part

Amplitude-phase form: $\boxed{y(t) = A\cos(\omega t - \varphi)}$

$A \geq 0$ real number called the amplitude

φ real number called the phase

We now explain how to convert between the 3 forms.

To express the complex form as a linear combination we write $c = \alpha + i\beta$ and compute

$$\operatorname{Re}(c e^{i\omega t}) = \operatorname{Re}((\alpha + i\beta)(\cos(\omega t) + i\sin(\omega t)))$$

$= \alpha\cos(\omega t) - \beta\sin(\omega t)$. We get

$\boxed{C_1 = \alpha,\ C_2 = -\beta}$, i.e. $\boxed{c = C_1 - iC_2}$, or

$\boxed{\bar{c} = C_1 + iC_2}$ where $\bar{c}$ is the complex conjugate of c

To express the <u>amplitude-phase form</u> as a linear combination, we use trigonometric formulas:

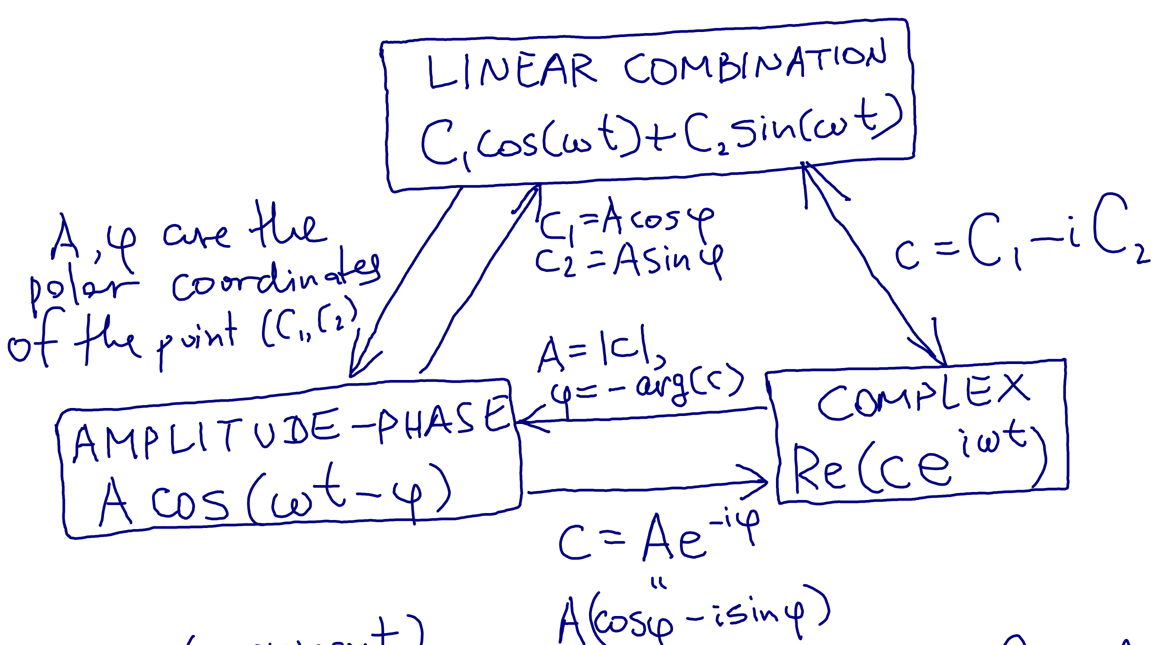
$$A\cos(\omega t-\varphi)=A(\cos\varphi\cdot\cos(\omega t)+\sin\varphi\cdot\sin(\omega t)).$$

We get $\boxed{C_1=A\cos\varphi,\quad C_2=A\sin\varphi}$

Together with the relation $\bar{c}=C_1+iC_2$ before this gives us a way to convert the amplitude-phase form to the complex form:

$$\boxed{\bar{c}=Ae^{i\varphi}} \quad\text{or}\quad c=Ae^{-i\varphi}$$

DIAGRAM for converting between 3 forms $\boxed{\text{TECHNIQUE}}$



Here $\arg(c)$ (argument) is the polar angle of the point on the complex plane corresponding to c.

TECHNIQUE PRACTICE

Example Find the amplitude-phase representation of the sinusoidal function

$y(t) = -\cos 2t + \sqrt{3}\sin 2t$

Solution We have $\boxed{\omega = 2}$. (because $2t\ldots$)

Step 1: write the general amplitude-phase form:

$A\cos(2t-\varphi)$.

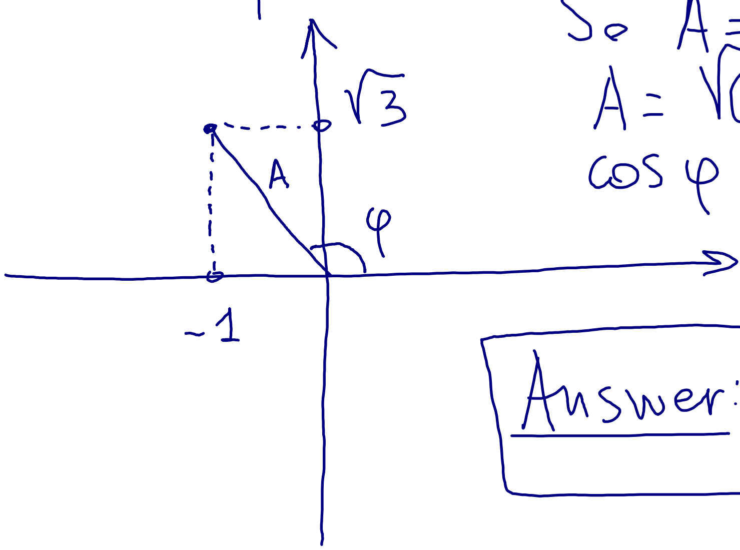
Step 2: express the general sinusoidal form as a linear combination:

$A\cos(2t-\varphi) = A\cos\varphi\cos(2t) + A\sin\varphi\sin(2t)$

Step 3: find A, φ s.t. $y(t) = A\cos(2t-\varphi)$

We have $\begin{cases} A\cos\varphi = -1 \\ A\sin\varphi = \sqrt{3} \end{cases}$ ← equated coefficients of $\cos(2t)$, $\sin(2t)$

This means that A, φ are the polar coordinates of the point $(-1, \sqrt{3})$:



So A = distance from 0 to $(-1, \sqrt{3})$

$A = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\cos\varphi = -1/2$, $\sin\varphi = \sqrt{3}/2$

$\varphi = 2\pi/3$

$\boxed{\textbf{Answer: } y(t) = 2\cos\left(2t - \frac{2\pi}{3}\right)}$

§4.3.2. Graphing sinusoidal functions

This is easiest done for a function in amplitude-phase form so it is suggested to convert into that form first. Then it is easy to find zeroes and max/min of the function

TECHNIQUE

Example: $-\cos(2t) + \sqrt{3}\sin(2t)$

Write it as $2\cos\left(2t - \frac{2\pi}{3}\right)$

Zeroes: $\cos\left(2t - \frac{2\pi}{3}\right) = 0 \iff 2t - \frac{2\pi}{3} = \frac{\pi}{2} + \pi k$

$\iff t = \frac{7\pi}{12} + \frac{\pi k}{2}$

Distance between 2 consecutive zeroes is $\frac{\pi}{2}$

Maxima: $\cos\left(2t - \frac{2\pi}{3}\right) = 1 \iff 2t - \frac{2\pi}{3} = 2\pi k$

$\iff t = \frac{\pi}{3} + \pi k.$

Value at the maxima is 2

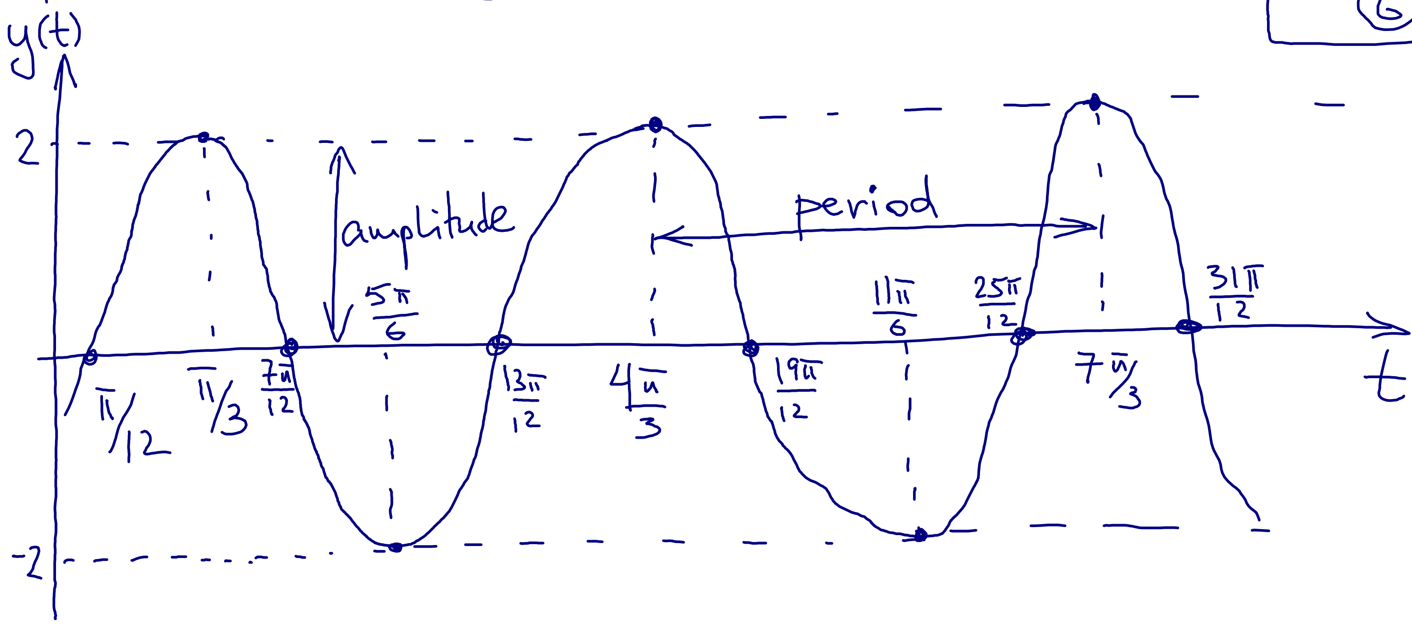
Distance between the maxima is π.

Minima: $\cos\left(2t - \frac{2\pi}{3}\right) = -1 \iff 2t - \frac{2\pi}{3} = \pi + 2\pi k$

$\iff t = \frac{5\pi}{6} + \pi k.$

Value at the minima is -2.

Here's the graph:



(for a prettier sinusoid, ask a computer...)

General facts about graphs of sinusoidal functions:

$\boxed{\text{TECHNIQUE}}$

- Distance between consecutive zeroes $= \frac{T}{2} = \frac{\pi}{\omega}$ where $T = \frac{2\pi}{\omega}$ is the period
- Distance between consecutive maxima $= T$, same for minima
- Value at maxima = the amplitude A
  Value at minima $= -A$
- Maxima/minima are at the midpoints between consecutive zeroes, e.g. on the graph above

$$\underset{\text{(maximum pt.)}}{\pi/3} = \frac{1}{2}\left(\pi/12 + 7\pi/12\right)$$

zeroes

This information lets us recover A, ω, φ from the graph of the function:

- A = maximal value
- To get ω, compute the period
- To get φ, plug in a maximal point

See RS 5, problem 5