# §4.3. Sinusoidal functions

We encountered in §4.2
functions of the form
$$e^{pt}(C_1 \cos(\omega t) + C_2 \sin(\omega t))$$
where $p, \omega, C_1, C_2$ are real numbers, $\omega > 0$

A special case with $p=0$ is called a sinusoidal function:

THEORY

Definition A sinusoidal function is a function of the form

$$\boxed{y(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)}$$

where $C_1, C_2, \omega$ are real constants and $\omega > 0$.

We call $\omega$ the frequency of this function.

The period is defined to be $\boxed{T = \frac{2\pi}{\omega}}$

Note that $y$ is $T$-periodic:
$$y(t+T) = y(t).$$

## §4.3.1. The three forms of sinusoidal functions

THEORY

A sinusoidal function can be written in the following 3 equivalent forms:

Linear combination: $\boxed{y(t) = C_1\cos(\omega t) + C_2\sin(\omega t)}$

$C_1, C_2$ real numbers

Complex form: $\boxed{y(t) = \mathrm{Re}(c e^{i\omega t})}$

$c$ complex number, Re = real part

Amplitude-phase form: $\boxed{y(t) = A\cos(\omega t - \varphi)}$

$A \geq 0$ real number called the amplitude

$\varphi$ real number called the phase

We now explain how to convert between the 3 forms.

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To express the complex form as a linear combination we write $c = \alpha + i\beta$ and compute

$$\mathrm{Re}(ce^{i\omega t}) = \mathrm{Re}((\alpha + i\beta)(\cos(\omega t) + i\sin(\omega t))$$

$$= \alpha\cos(\omega t) - \beta\sin(\omega t).$$ We get

$\boxed{C_1 = \alpha,\ C_2 = -\beta}$, i.e. $\boxed{c = C_1 - iC_2}$, or

$\boxed{\bar{c} = C_1 + iC_2}$ where $\bar{c}$ is the complex conjugate of $c$

To express the amplitude-phase form as a linear combination, we use trigonometric formulas:

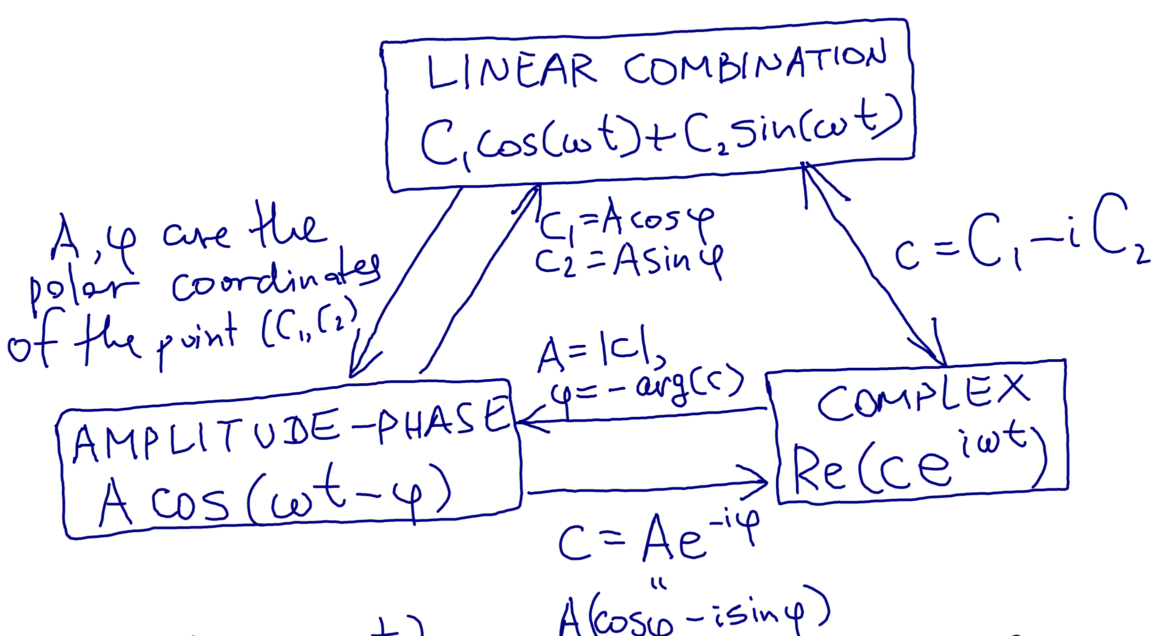
$$A\cos(\omega t-\varphi)=A(\cos\varphi\cdot\cos(\omega t)+\sin\varphi\cdot\sin(\omega t)).$$

We get $\boxed{C_1=A\cos\varphi,\quad C_2=A\sin\varphi}$

Together with the relation $\bar{c}=C_1+iC_2$ before this gives us a way to convert the amplitude-phase form to the complex form:

$$\boxed{\bar{c}=Ae^{i\varphi}} \quad \text{or} \quad c=Ae^{-i\varphi}$$

## DIAGRAM for converting between 3 forms $\boxed{\text{TECHNIQUE}}$




Here $\arg(c)$ (argument) is the polar angle of the point on the complex plane corresponding to $c$.

TECHNIQUE PRACTICE

**Example** Find the amplitude-phase representation of the sinusoidal function

$y(t) = -\cos 2t + \sqrt{3}\sin 2t$

**Solution** We have $\boxed{\omega = 2}$. (because $2t \ldots$)

**Step 1:** write the general amplitude-phase form:

$A\cos(2t - \varphi)$.

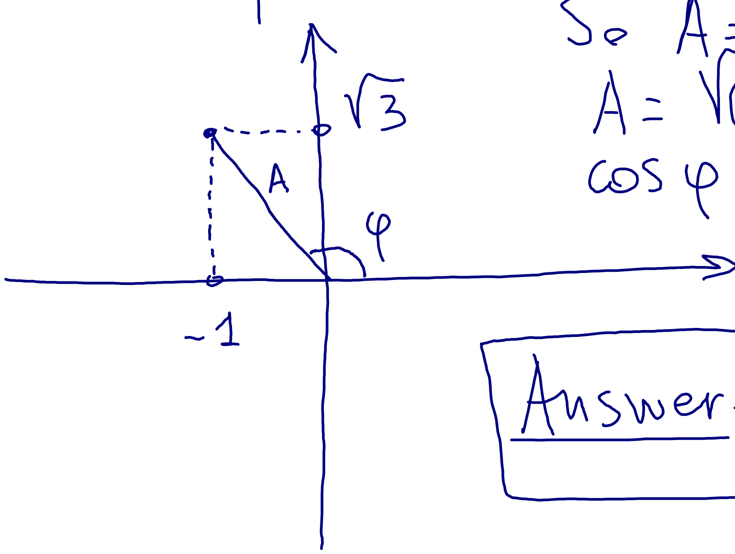
**Step 2:** express the general sinusoidal form as a linear combination:

$A\cos(2t-\varphi) = A\cos\varphi\cos(2t) + A\sin\varphi\sin(2t)$

**Step 3:** find $A, \varphi$ s.t. $y(t) = A\cos(2t-\varphi)$

We have $\begin{cases} A\cos\varphi = -1 \\ A\sin\varphi = \sqrt{3} \end{cases}$ ← equated coefficients of $\cos(2t)$, $\sin(2t)$

This means that $A, \varphi$ are the polar coordinates of the point $(-1, \sqrt{3})$:




So $A$ = distance from $O$ to $(-1, \sqrt{3})$

$A = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\cos\varphi = -1/2$, $\sin\varphi = \sqrt{3}/2$

$\varphi = 2\pi/3$

$$\boxed{\textbf{Answer: } y(t) = 2\cos\left(2t - \frac{2\pi}{3}\right)}$$

# §4.3.2. Graphing sinusoidal functions

TECHNIQUE

This is easiest done for a function in amplitude-phase form so it is suggested to convert into that form first. Then it is easy to find