Here we study equations of the form

\[ a_2 y''(t) + a_1 y'(t) + a_0 y(t) = 0 \]  

(\star)

where \( a_0, a_1, a_2 \) are real constants and \( a_2 \neq 0 \). We are looking for the general solution to (\star) and our final answer should only involve real valued functions (no complex stuff).

From §4.1.3 we know that it is enough to find two linearly independent solutions \( y_1(t), y_2(t) \) to (\star), then the general solution is

\[ y(t) = C_1 y_1(t) + C_2 y_2(t). \]

§4.2.1. Exponential substitution & characteristic equation

We will look for solutions to (\star) in the form

\[ y(t) = e^{zt} \]

where \( z \) is some constant (possibly complex).

Let us substitute into the equation and use that \( y'(t) = z e^{zt} \)
\[ y''(t) = z^2 e^{zt}. \]
\[ a_2 y''(t) + a_1 y'(t) + a_0 y(t) = 0 \]
\[ a_2 z^2 e^{zt} + a_1 z e^{zt} + a_0 e^{zt} = 0 \]

That is, \((a_2 z^2 + a_1 z + a_0) e^{zt} = 0\).

Which just means \(a_2 z^2 + a_1 z + a_0 = 0\).

But that is a polynomial equation on \(z\)
(no dependence on \(t\)!)

**Theorem**

The function

\[ y(t) = e^{zt} \quad (z = \text{const}) \]

solves the equation

\[ a_2 y'' + a_1 y' + a_0 y = 0 \quad (*) \]

if and only if \(z\) solves the quadratic equation

\[ a_2 z^2 + a_1 z + a_0 = 0. \quad (∇) \]

Equation (∇) is called the characteristic equation corresponding to (*).

And its roots \(z\) are called the characteristic roots of the ODE (*).

Now we have 3 cases depending on what kind of roots (∇) has:
Case 1: $(\forall)$ has two distinct real roots, let's call them $z_1$ and $z_2$. Then $e^{z_1 t}$, $e^{z_2 t}$ are two linearly independent solutions to $(\times)$. [I will not prove linear independence of these and you won't be asked to do it either.]

Example 1: $y'' - y = 0$

Characteristic equation: $z^2 - 1 = 0$

Characteristic roots: $z_1 = 1$, $z_2 = -1$

2 linearly independent solutions: $y_1(t) = e^t$, $y_2(t) = e^{-t}$

General solution: $y(t) = c_1 e^t + c_2 e^{-t}$

Case 2: $(\forall)$ has a double (real) root, let's call it $z_1$. Then $e^{z_1 t}$ is a solution to $(\times)$ but where to get the other solution? Turns out that $t e^{z_1 t}$ is also a solution, and then $e^{z_1 t}$, $t e^{z_1 t}$ are two lin. ind. solutions.

Proof of the fact that $t e^{z_1 t}$ solves $(\times)$ (in case you were curious; you can certainly skip this)
\[ y = te^{zt} \]
\[ y' = e^{zt} + z_te^{zt} \]
\[ y'' = 2z_te^{zt} + z_t^2e^{zt} \]

Then \( a_2y'' + a_1y' + a_0y \)
\[ = (a_2z_t^2 + a_1z_t + a_0)te^{zt} + (2a_2z_t + a_1)e^{zt} \]

Since \( z_t \) is a characteristic root

Now remember that \( z_t \) is a double root. This means that \( D = a_1^2 - 4a_0a_2 = 0 \) and

\[ z_1 = -\frac{a_1}{2a_2} \]

So \( 2a_2z_1 + a_1 = 0 \).

It follows that \( a_2y'' + a_1y' + a_0y = 0 \). \( \square \)

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**Example 2:** \[ y'' = 0 \]

**Characteristic equation:** \( z^2 = 0 \)

**Characteristic roots:** \( z_1 = 0 \) double root

2 linearly independent solutions: \( y_1(t) = 1, \ y_2(t) = t \)

**General solution:** \[ y(t) = C_1 + C_2t \]

**Example 3:** \[ y'' + 4y' + 4y = 0 \]

**Ch. eqn:** \( z^2 + 4z + 4 = 0 \) \( \text{Ch. roots: } z_t = -2 \)

**General solution:** \( C_1e^{-2t} + C_2te^{-2t} \)
Case 3: $(v)$ has two complex roots $z_1 = p + iq$, $z_2 = p - iq$

(here $p = -\frac{a_1}{2a_2}$, $q = \frac{\sqrt{D}}{2a_2}$, $D = a_1^2 - 4a_0a_2$)

We have two linearly independent solutions $e^{z_1t}$, $e^{z_2t}$ but they are complex valued.

How to get real valued ones?

Take the real and imaginary part of $e^{z_1t}$:

\[ y_1(t) = \text{Re} \left( e^{z_1t} \right) = \text{Re} \left( e^{(p+iq)t} \right) = e^{pt} \cos(qt) \]
\[ y_2(t) = \text{Im} \left( e^{z_1t} \right) = \text{Im} \left( e^{(p+iq)t} \right) = e^{pt} \sin(qt) \]

Then $y_1, y_2$ are linearly independent solutions to $(v)$.

What if we took $\text{Re} \left( e^{z_2t} \right)$, $\text{Im} \left( e^{z_2t} \right)$ instead?

Get basically the same functions:

\[ \text{Re} \left( e^{z_2t} \right) = e^{pt} \cos(qt), \quad \text{Im} \left( e^{z_2t} \right) = -e^{pt} \sin(qt) \]

Example 4: the harmonic oscillator

\[ my'' + ky = 0 \]

where $m > 0$, $k > 0$ are given parameters.

Characteristic equation: $mz^2 + k = 0$

Characteristic roots: $z_1 = i\omega$, $z_2 = -i\omega$

where $\omega := \sqrt{\frac{k}{m}}$. 

Greek letter omega
Complex exponential solutions: $e^{i\omega t}$, $e^{-i\omega t}$

Real solutions: $y_1 = \cos(\omega t)$, $y_2 = \sin(\omega t)$

General solution: $y = C_1 \cos(\omega t) + C_2 \sin(\omega t)$

Note: $e^{i\omega t}$ are linear combinations of $\cos(\omega t)$, $\sin(\omega t)$ with complex coefficients.

$e^{\pm i\omega t} = \cos(\omega t) \pm i \sin(\omega t)$

and conversely, $\cos(\omega t)$ and $\sin(\omega t)$ are linear combinations of $e^{i\omega t}$, $e^{-i\omega t}$:

$\cos(\omega t) = \frac{1}{2} (e^{i\omega t} + e^{-i\omega t})$

$\sin(\omega t) = \frac{1}{2i} (e^{i\omega t} - e^{-i\omega t})$

§4.2.2. The general algorithm

Here is the algorithm for finding the general solution to the ODE $a_2 y'' + a_1 y' + a_0 y = 0$

Step 1: Write the characteristic equation

$a_2 z^2 + a_1 z + a_0 = 0$

Step 2: Find the roots of the characteristic equation (using the quadratic formula)

Recall the discriminant $D = a_1^2 - 4a_0a_2$
Step 3: Write the general solution using the following 3 cases for the roots:

Case 1: 2 distinct real roots $z_1, z_2$:
$$y(t) = C_1 e^{z_1 t} + C_2 e^{z_2 t}$$

Case 2: 1 double root $z_1$:
$$y(t) = C_1 e^{z_1 t} + C_2 t e^{z_1 t}$$

Case 3: 2 complex roots $z_1 = p + iq$, $z_2 = p - iq$:
$$y(t) = C_1 e^{pt} \cos(qt) + C_2 e^{pt} \sin(qt)$$

In practice ("on the test") knowing this algorithm is more important than remembering how we arrived to it (§§4.1, 4.2.1)

§4.2.3 Example: damped harmonic oscillator (dashpot)

Let us add resistance to the spring model we had before:

- $m > 0$ mass
- $k > 0$ spring constant

![Diagram of damped harmonic oscillator]
Recall that 
\[ t = \text{time (say, in seconds)} \]
\[ y(t) = \text{displacement of the mass from the equilibrium position (say, in meters)} \]

**Newton’s II nd Law:**
\[ m \cdot y''(t) = \text{total force} = F_{\text{spring}} + F_{\text{dashpot}} \]

**Hooke’s Law:** \[ F_{\text{spring}} = -k \cdot y(t) \]

We’ll model the dashpot by putting 
\[ F_{\text{dashpot}} = -b \cdot y'(t) \]

(The faster we move, the higher the damping force)

Putting these together we get the **damped harmonic oscillator ODE:**
\[ m \cdot y'' + b \cdot y' + k \cdot y = 0 \]

We will find the **general solution** and study its **qualitative behavior**
Characteristic equation

\[ m z^2 + b z + k = 0 \]

Characteristic roots

\[ z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \]

3 Cases:

Case 1: \( b^2 > 4mk \) (overdamped)

\[ z_1 = \frac{-b + \sqrt{b^2 - 4mk}}{2m}, \quad z_2 = \frac{-b - \sqrt{b^2 - 4mk}}{2m} \]

General solution: \( y(t) = c_1 e^{z_1 t} + c_2 e^{z_2 t} \)

Case 2: \( b^2 = 4mk \) (critically damped)

\[ z_1 = -\frac{b}{2m}, \quad \text{double root} \]

General solution: \( y(t) = c_1 e^{z_1 t} + c_2 te^{z_1 t} \)

Case 3: \( b^2 < 4mk \) (underdamped)

\[ z_{1,2} = \frac{-b \pm \sqrt{4mk - b^2}}{2m} = P \pm iq, \quad \text{where} \]

\[ P = -\frac{b}{2m}, \quad q = \sqrt{\frac{k}{m} - P^2}. \]

General solution: \( y(t) = c_1 e^{P t} \cos(qt) + c_2 e^{P t} \sin(qt) \)
Now for qualitative behavior:

- If \( b = 0 \) (no damping: the usual harmonic oscillator)

then we are in Case 3 and

\[
y(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)
\]

where \( \omega = \sqrt{\frac{k}{m}} \) is called the undamped frequency.

Functions of this form are called sinusoidal functions (more in §4.3) and they oscillate at frequency \( \omega \).

- Now let us assume that \( b > 0 \).

Then in all 3 cases \( y(t) \) is exponentially decaying:

**Case 3:** decay rate \( \frac{b}{2m} \). We write

\[
y(t) = e^{-\frac{bt}{2m}} (C_1 \cos(\omega t) + C_2 \sin(\omega t))
\]

Sinusoidal function with frequency \( \omega \).

**Case 2:** decay rate \( \frac{b}{2m} \) (if we forget about the factor of \( t \))

no oscillation

**Case 1:** decay rate \(-\frac{b}{2m}\) (as \( e^{\frac{b}{2m}t} \) decays even faster than \( e^{\frac{b}{2m}t} \) as \( \frac{b}{2m} \) \( > \) \( 2 \omega \))

and no oscillation. Note: \( 0 < -\frac{b}{2m} < \frac{b}{2m} \) in Case 3

MORAL: too much damping \( \rightarrow \) slower return to equilibrium.