§4. Linear ODEs with constant coefficients

We will study equations of the form

$$a_k y^{(k)} + \cdots + a_1 y' + a_0 y = 0 \quad \text{(homogeneous)}$$

$$a_k y^{(k)} + \cdots + a_1 y' + a_0 y = b(t) \quad \text{(inhomogeneous)}$$

where $a_0, a_1, \ldots, a_k$ are constants.

We will focus on the case $k=2$: it is the one most common in applications.

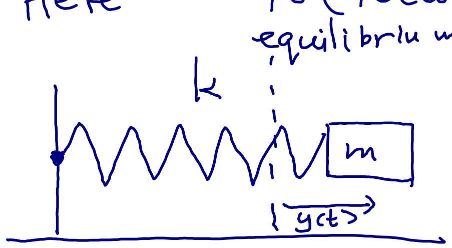
§4.1. Harmonic oscillator & Superposition principle

§4.1.1. Harmonic oscillator: modeling a spring

We have a spring of mass m (say, in kg) with coefficient k (in $N/\text{met} = \text{kg}/\text{sec}^2$) (will be used in Hooke's law below)

Here $N \text{(Newton)} = \dfrac{\text{kg} \cdot \text{meter}}{\text{Second}^2}$

THEORY



We are modeling the motion of the spring

Denote:

t = time (in seconds)

$y(t)$ = displacement from equilibrium at time t (in meters)

Note: $m > 0$, $k > 0$

To model the situation, we use:

- Newton's IInd Law:

(mass)·(acceleration) = force

The left-hand side of this is

$m \cdot y''(t)$ (in Newtons)

The force is given by

- Hooke's Law:

Spring force = −(Spring coefficient)·(displacement)

So the force is $-k \cdot y(t)$.

This gives the harmonic oscillator equation:

$$\boxed{m \cdot y'' + k \cdot y = 0}$$

This is a 2^{nd} order linear homogeneous ODE with constant coefficients

§4.1.2. Superposition Principle

For simplicity let us take $m=1, k=1$, i.e. we study the equation

THEORY

$$\boxed{y'' + y = 0.}$$

Can we guess some solutions?

$y = \cos t$
$y = \sin t$ } both solve the equation

Later in §4.2 we study a general procedure of obtaining these solutions

How to construct more solutions?

Theorem Any function of the form

$\boxed{\text{THEORY}}$

$$y(t) = C_1 \cdot \cos t + C_2 \cdot \sin t,$$

where C_1, C_2 are constants, solves the differential equation $y'' + y = 0$

Proof We compute

$$y'' = C_1 (\cos t)'' + C_2 \cdot (\sin t)''$$
$$= -C_1 \cos t - C_2 \sin t$$
$$y = C_1 \cos t + C_2 \sin t$$

So $y'' + y = 0$. $\square$

This is a special case of the superposition principle. To state it we use the general

Definition Assume that $y_1(t), y_2(t), \dots, y_k(t)$ are some functions. A *linear combination* of $y_1, \dots, y_k$ is a function of the form

$$c_1 y_1(t) + c_2 y_2(t) + \dots + c_k y_k(t)$$

where $c_1, \dots, c_k$ are some constants.

<u>Example:</u> here are some linear combinations of $y_1(t) = \cos t$ and $y_2(t) = \sin t$:

$y(t) = 3\cos t + 2\sin t$ $\quad (a_1 = 3,\ a_2 = 2)$

$y(t) = \sin t$ $\quad (a_1 = 0,\ a_2 = 1)$

$y(t) = 0$ $\quad (a_1 = 0,\ a_2 = 0)$

Theorem (Superposition Principle, homogeneous case)

Assume that the functions $y_1(t), \dots, y_n(t)$ all solve the linear homogeneous ODE

$$a_k(t)y^{(k)} + \dots + a_1(t)y' + a_0(t)y = 0. \quad (*)$$

Then any linear combination of $y_1, \dots, y_n$ also solves $(*)$.

<u>Proof</u> Similar to the previous theorem but more tedious... $\square$

<u>Remarks</u>

1. This is a big reason why linear equations are easy to solve: we can get a lot of solutions if we know just a few solutions.
2. For inhomogeneous equations we'll have another version later.
3. For nonlinear equations, the Superposition Principle fails. See Recitation Sheet 1, Problem 3(b)

§4.1.3. The general solution

Coming back to the harmonic oscillator

$$\boxed{y'' + y = 0}$$

, we know that linear combinations $C_1 \cos t + C_2 \sin t$ are solutions. It turns out