§4. Linear ODEs with constant coefficients

We will study equations of the form

$$a_k y^{(k)} + \cdots + a_1 y' + a_0 y = 0 \quad \text{(homogeneous)}$$

$$a_k y^{(k)} + \cdots + a_1 y' + a_0 y = b(t) \quad \text{(inhomogeneous)}$$

where $a_0, a_1, \ldots, a_k$ are <u>constants</u>.

We will focus on the case $k=2$: it is the one most common in applications.

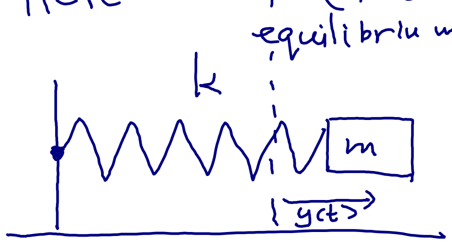
§4.1. Harmonic oscillator & Superposition principle

§4.1.1. Harmonic oscillator: modeling a spring

We have a spring of mass m (say, in kg) with coefficient k (in $N/\text{met} = \text{kg}/\text{sec}^2$)
(will be used in Hooke's law below)

Here $N \text{(Newton)} = \dfrac{\text{kg} \cdot \text{meter}}{\text{Second}^2}$

$\boxed{\text{THEORY}}$



We are modeling the motion of the spring

<u>Denote</u>:

t = time (in seconds)

$y(t)$ = displacement from equilibrium at time t (in meters)

<u>Note</u>: $m > 0$, $k > 0$

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To model the situation, we use:

- Newton's IInd Law:

(mass)·(acceleration) = force

The left-hand side of this is

$m \cdot y''(t)$ (in Newtons)

The force is given by

- Hooke's Law:

Spring force = −(Spring coefficient)·(displacement)

So the force is $-k \cdot y(t)$.

This gives the harmonic oscillator equation:

$$\boxed{m \cdot y'' + k \cdot y = 0}$$

This is a 2nd order linear homogeneous ODE with constant coefficients

§4.1.2. Superposition Principle

For simplicity let us take $m=1, k=1$, i.e. we study the equation

THEORY

$$\boxed{y'' + y = 0.}$$

Can we guess some solutions?

$y = \cos t$
$y = \sin t$
both solve the equation

Later in §4.2 we study a general procedure of obtaining these solutions

How to construct more solutions?

<u>Theorem</u> Any function of the form $\boxed{\text{THEORY}}$

$$y(t) = C_1 \cdot \cos t + C_2 \cdot \sin t,$$

where C_1, C_2 are constants, solves the differential equation $y'' + y = 0$

<u>Proof</u> We compute

$$y'' = C_1 \cdot (\cos t)'' + C_2 \cdot (\sin t)''$$
$$= -C_1 \cos t - C_2 \sin t$$
$$y = C_1 \cos t + C_2 \sin t$$

So $y'' + y = 0$. $\square$

This is a special case of the <u>superposition principle</u>. To state it we use the general

<u>Definition</u> Assume that $y_1(t), y_2(t), \dots, y_k(t)$ are some functions. A <u>linear combination</u> of $y_1, \dots, y_k$ is a function of the form

$$c_1 y_1(t) + c_2 y_2(t) + \dots + c_k y_k(t)$$

where $c_1, \dots, c_k$ are some constants.

Example: here are some linear combinations of $y_1(t) = \cos t$ and $y_2(t) = \sin t$:

$y(t) = 3\cos t + 2\sin t$ $\quad (a_1 = 3, a_2 = 2)$

$y(t) = \sin t$ $\quad (a_1 = 0, a_2 = 1)$

$y(t) = 0$ $\quad (a_1 = 0, a_2 = 0)$

Theorem (Superposition Principle, homogeneous case)

Assume that the functions $y_1(t), \ldots, y_n(t)$ all solve the linear homogeneous ODE

$$a_k(t) y^{(k)} + \cdots + a_1(t) y' + a_0(t) y = 0. \quad (*)$$

Then any linear combination of $y_1, \ldots, y_n$ also solves $(*)$.

Proof Similar to the previous theorem but more tedious... $\square$

Remarks

1. This is a big reason why linear equations are easy to solve: we can get a lot of solutions if we know just a few solutions.
2. For inhomogeneous equations we'll have another version later.
3. For nonlinear equations, the Superposition Principle fails. See Recitation Sheet 1, Problem 3(b) for an example.

§4.1.3. The general solution

Coming back to the harmonic oscillator $\boxed{y''+y=0}$, we know that linear combinations $C_1\cos t + C_2 \sin t$ are solutions. It turns out that there are no other solutions: THEORY

<u>Theorem</u> The general solution to the equation $y''+y=0$ is given by $y=C_1\cos t + C_2\sin t$ where C_1, C_2 are arbitrary constants. □

<u>Proof</u> Way too complicated for our course...

This is a special case of a general statement. To formulate it we need

<u>Definition</u> Let $y_1(t), \dots, y_k(t)$ be some functions.

- A <u>nontrivial linear combination</u> of $y_1, \dots, y_k$ is a function of the form $c_1 y_1(t) + \dots + c_k y_k(t)$ where $c_1, \dots, c_k$ are constants and at least one of these constants is $\neq 0$
- We say $y_1, \dots, y_k$ are <u>linearly dependent</u>, if there exists a nontrivial linear combination $c_1 y_1(t) + \dots + c_k y_k(t)$ which is equal to 0 for all t;

- linearly independent otherwise, i.e. each nontrivial linear combination of $y_1,\dots,y_k$ is not identically 0.

Examples: 1 $y_1(t)=\cos t$, $y_2(t)=2\cos t$ are linearly dependent: $-2y_1+y_2=0$ for all t

2 $y_1(t)=\cos t$, $y_2(t)=\sin t$ are linearly independent.

Proof Assume that for some constants c_1, c_2,
$c_1\cos t+c_2\sin t=0$ for all t.
Substitute $t=0$: get $c_1\cdot 1+c_2\cdot 0=0 \Rightarrow \boxed{c_1=0}$
$t=\pi/2$: get $c_1\cdot 0+c_2\cdot 1=0 \Rightarrow \boxed{c_2=0}$

So the only linear combination of y_1, y_2 which is identically 0 is the trivial one. □

Remark: two functions y_1, y_2 are linearly dependent $\Leftrightarrow$ one of them is a constant multiple of the other one.
For more than 2 functions this does not work as shown by the example below:

3 $y_1(t)=\cos t$, $y_2(t)=\sin t$, $y_3(t)=\sin(t+\pi/4)$ are linearly dependent: $y_3=\frac{\sqrt2}{2}y_1+\frac{\sqrt2}{2}y_2$,

So $-\frac{\sqrt2}{2}y_1-\frac{\sqrt2}{2}y_2+y_3=0$.

Theorem (Form of general solution to homogeneous linear ODE)

Assume that $y_1(t), \dots, y_k(t)$ are linearly independent solutions to the ODE

$$a_k(t)\,y^{(k)}(t) + \cdots + a_1(t)\,y'(t) + a_0(t)\,y(t) = 0. \quad (*)$$

Then the general solution to $(*)$ has the form

$$c_1 y_1(t) + \cdots + c_k y_k(t) \quad \text{where}$$

$\boxed{\text{THEORY}}$

$c_1, \dots, c_k$ are arbitrary constants.

Important: for this to be true, the number of functions $y_1, \dots, y_k$ has to be equal to the order of the ODE. E.g.: for 2^{nd} order equations, need to find 2 linearly independent solutions.

"ALGORITHM" for solving a linear homogeneous ODE:

- Find k linearly independent solutions $y_1(t), \dots, y_k(t)$ of the ODE where $k =$ the order of the ODE. (This is the hard part, we learn how to do this next time)
- The general solution is $c_1 y_1(t) + \cdots + c_k y_k(t)$
- If solving the initial value problem, use k initial conditions to find $c_1, \dots, c_k$.

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For 2^{nd} order equations initial conditions have the form

$$\begin{cases} y(t_0) = A \\ y'(t_0) = B \end{cases}$$

← need to specify the value of y and of its 1^{st} derivative

Example: solve the problem

$$\begin{cases} y'' + y = 0 \\ y(\pi/4) = 1 \\ y'(\pi/4) = 0 \end{cases}$$

← initial position

← initial velocity

We took the initial time $t_0 = \pi/4$ to make things more fun...

Solution
- 2 linearly independent solutions:

$$y_1 = \cos t, \quad y_2 = \sin t$$

- General solution: $y = C_1 \cos t + C_2 \sin t$
- Plug in the initial conditions & use $y' = -C_1 \sin t + C_2 \cos t$

$$\begin{cases} 1 = y(\pi/4) = C_1 \cos \pi/4 + C_2 \sin \pi/4 = (C_1 + C_2) \cdot \sqrt{2}/2 \\ 0 = y'(\pi/4) = -C_1 \sin \pi/4 + C_2 \cos \pi/4 = (-C_1 + C_2) \cdot \sqrt{2}/2 \end{cases}$$

- Solve for C_1, C_2:

$$\begin{cases} C_1 + C_2 = \sqrt{2} \\ -C_1 + C_2 = 0 \end{cases} \Rightarrow C_2 = C_1$$

$$\Longrightarrow 2C_1 = \sqrt{2} \Rightarrow C_1 = \frac{\sqrt{2}}{2}$$

$$\Longrightarrow C_2 = \frac{\sqrt{2}}{2}$$

The answer is

$$\boxed{y(t) = \frac{\sqrt{2}}{2} \cos t + \frac{\sqrt{2}}{2} \sin t}$$

← a sinusoidal function (see §4.3 later)