§3.2. The complex exponential

THEORY

We know about exponentials of real numbers:

real number $a \mapsto$ real number e^a

and we know that $\boxed{e^{a+b} = e^a \cdot e^b}$
(multiplicative property)

Now we will learn how to take exponentials of complex numbers.

The key is Euler's formula:

for any real number b,

$$\boxed{e^{b\cdot i} = \cos b + i \cdot \sin b}$$

Why is it true? One can "prove" this using Taylor series (we won't do it here) but we instead just use the above as the definition of $e^{b\cdot i}$

What is great is that the multiplicative property still holds: for any real b, c

$$\boxed{e^{(b+c)i} = e^{bi} \cdot e^{ci}}.$$

Proof: we write

$$e^{(b+c)i} = \cos(b+c) + i\sin(b+c)$$
$$e^{bi} = \cos b + i\sin b$$
$$e^{ci} = \cos c + i\sin c$$

$$e^{bi}\cdot e^{ci} = (\cos b\cdot\cos c - \sin b\cdot\sin c) + (\cos b\cdot\sin c + \sin b\cdot\cos c)i$$

It remains to use the formulas for $\cos(b+c)$ and $\sin(b+c)$. $\square$

We also get neat identities

$$\boxed{e^{\pi i} = -1, \quad e^{2\pi i} = 1}$$

And we can express $\cos b$, $\sin b$ via e^{ib}:

$$\boxed{\begin{aligned} \cos b &= \mathrm{Re}(e^{ib}) = \frac{e^{ib}+e^{-ib}}{2} \\ \sin b &= \mathrm{Im}(e^{ib}) = \frac{e^{ib}-e^{-ib}}{2i} \end{aligned}}$$

← note the relation to hyperbolic sine/cosine

Now for a general complex number $z = a+ib$ we define

$$\boxed{e^z = e^a\cdot e^{ib} = e^a\cos b + e^a\sin b\cdot i}$$

and we still have the multiplicative property

$$\boxed{e^{z+w} = e^z\cdot e^w}$$ for any two complex numbers z, w

Differentiating complex exponentials

The main reason we care about exponentials (real or complex) in this course is that they solve a fundamental differential equation:

THEORY

Theorem Fix a complex number $z = a + bi$ and define the complex valued function of a real argument t

$$f(t) = e^{z \cdot t} = e^{at}(\cos(bt) + i\sin(bt)).$$

Then $f(t)$ solves the differential equation

$$\boxed{f'(t) = z \cdot f(t),} \quad \text{i.e.} \quad \boxed{\frac{d}{dt} e^{z \cdot t} = z \cdot e^{z \cdot t}}$$

Here $f'(t)$ is defined by differentiating the real and imaginary parts.

Proof Case 1: $z = a$, $f(t) = e^{at}$

$$f'(t) = a e^{at}$$

Case 2: $z = ib$, $f(t) = e^{ibt} = \cos(bt) + i\sin(bt)$

$$f'(t) = -b\sin(bt) + ib\cos(bt)$$

$$ibf(t) = ib\cos(bt) - b\sin(bt)$$

So $f'(t) = z f(t)$

General case: Similar but more tedious... $\square$

Polar form

Each $z = a + ib$ can be written in polar coordinates:

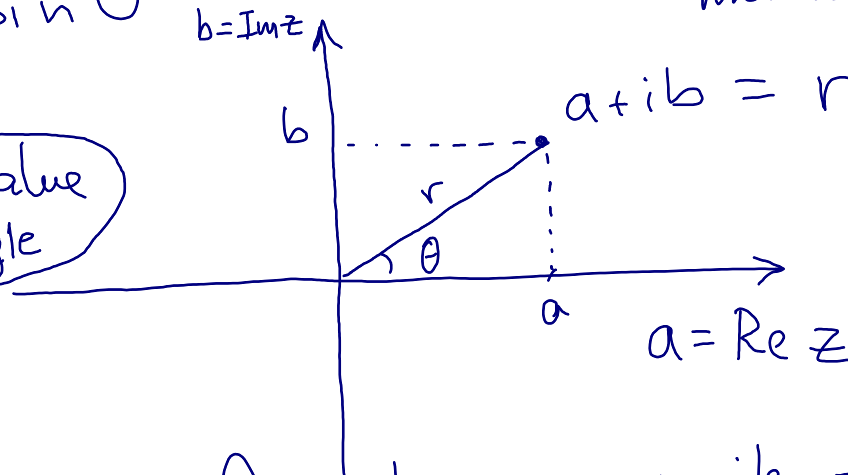
Greek letter theta

$$\begin{cases} a = r\cos\theta \\ b = r\sin\theta \end{cases}$$

where $r \geq 0$, θ real number
θ defined uniquely modulo $+2\pi k$, k integer

$a + ib = re^{i\theta}$

r = absolute value
θ = polar angle



Using Euler's formula we write this as

$$z = re^{i\theta}$$

Multiplication is easy in polar form:

if $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ then

$$z_1 \cdot z_2 = r_1 \cdot r_2\, e^{i(\theta_1 + \theta_2)}$$

"To multiply two complex numbers, multiply the absolute values and add the polar angles".

On the other hand, addition is difficult in polar form...

Polar form is useful for finding roots:

Example: Find all complex solutions to the equation $z^3+1=0$
(note: we did this also at the end of §3.1)

Solution: we rewrite the equation as

$$z^3 = -1.$$

Write in polar form: $z = re^{i\theta}$ $(r>0)$, $-1 = e^{i\pi}$

Then $z^3 = r^3 e^{3i\theta} = e^{i\pi}$

Comparing absolute values, we see that $\boxed{r=1}$.

Now, $e^{3i\theta} = e^{i\pi}$, meaning that

$3\theta = \pi + 2\pi k$ for some integer k.

(because the point z^3 has polar angles 3θ and π and polar angle is