§3. Complex numbers

In calculus, we typically work with real numbers.
Complex numbers are an extension of real numbers obtained by adding the imaginary unit

$$i = \sqrt{-1}$$

($\sqrt{-1}$ does not exist in real numbers: the equation $x^2 = -1$ has no real solutions)
They turn out to have a lot of amazing properties, some of which we will use in this course.

§3.1. Basic algebraic properties

THEORY

Definition A complex number is an expression of the form

$a + bi$ where a, b are real numbers and i is the imaginary unit

Examples:

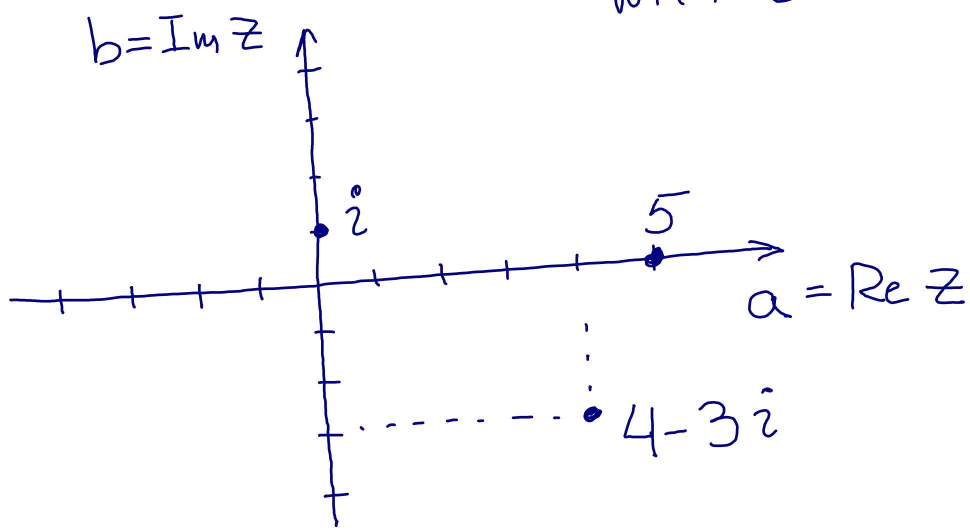
- Any real a is a complex number: $a = a + 0\cdot i$, e.g. $5 = 5 + 0\cdot i$
- $i = 0 + 1\cdot i$ is a complex number

For $z = a + bi$
we say that
a is the real part of z
b is the imaginary part of z
and write $\boxed{\operatorname{Re} z = a, \quad \operatorname{Im} z = b}$

Example: $\operatorname{Re} 5 = 5$, $\operatorname{Im} 5 = 0$
$\operatorname{Re} i = 0$, $\operatorname{Im} i = 1$

Graphic representation:

complex number $a+ib$ $\to$ point on the plane with coordinates (a,b)



Algebraic operations

Addition/subtraction:
$$(a+bi) + (c+di) = (a+c) + (b+d)i$$
$$(a+bi) - (c+di) = (a-c) + (b-d)i$$

Example:

$(1+3i)-(2+i) = -1+2i$

Multiplication: use that $i^2 = -1$

$(a+bi)(c+di) = ac + bci + adi + bdi^2$

$= (ac-bd) + (bc+ad)i$

Examples:

- $(2+3i)(3-i) = 6+9i-2i-3i^2$
  $= 9+7i$
- $(1+i)^2 = 2i$

Complex conjugate: if $z = a+bi$ then the complex conjugate is

$\bar{z} = a - bi$

Example: $\overline{2+3i} = 2-3i$

Absolute value: if $z = a+bi$ then $|z| = \sqrt{a^2+b^2}$

Useful identity: $\boxed{z \cdot \bar{z} = |z|^2}$

Proof: $(a+bi)(a-bi) = a^2-b^2$.

More identities: $\overline{z+w} = \bar{z}+\bar{w}$, $\overline{zw} = \bar{z}\,\bar{w}$

$|zw| = |z| \cdot |w|$

Dividing complex numbers:

if z, w are complex, $w \neq 0$

then compute $\frac{z}{w} = \frac{z \cdot \bar{w}}{w \cdot \bar{w}} = \frac{z \cdot \bar{w}}{|w|^2}$

multiply numerator & denominator by $\bar{w}$

$|w|^2$ is a real number, easy to divide

Example: $\frac{1+2i}{1+i} = \frac{(1+2i)(1-i)}{(1+i)(1-i)}$

$= \frac{3+i}{2} = \frac{3}{2} + \frac{1}{2} i.$

Solving quadratic equations

$az^2 + bz + c = 0$

where a, b, c are real, $a \neq 0$

Quadratic formula for the roots:

$$z_{1,2} = \frac{-b \pm \sqrt{D}}{2a} \quad \text{where } D := b^2 - 4ac$$

3 cases:

$D > 0$: 2 real roots

$D = 0$: 1 root only, of multiplicity 2 (double root)

$D < 0$: 2 complex roots $z_{1,2} = \frac{-b \pm \sqrt{-D} \cdot i}{2a}$

Examples:

1. $2z^2 + z + 1 = 0$

$a = 2,\ b = 1,\ c = 1,\ D = b^2 - 4ac = -7 < 0$

The roots are

$$z_1 = \frac{-1 + \sqrt{7}\,i}{4}, \quad z_2 = \frac{-1 - \sqrt{7}\,i}{4}.$$

Note: $z_2 = \bar{z}_1$

2. $z^2 - 5z + 6 = 0$

$a = 1,\ b = -5,\ c = 6,\ D = 1 > 0$

The roots are

$$z_1 = \frac{5+1}{2} = 3, \quad z_2 = \frac{5-1}{2} = 2$$

3. $z^2 + 4z + 4 = 0$

$a = 1,\ b = 4,\ c = 4,\ D = 0$

Only one root: $z = \frac{-4}{2} = -2$

Factorization:

If z_1, z_2 are the roots of the quadratic equation $az^2 + bz + c = 0$ then

$$az^2 + bz + c = a(z - z_1)(z - z_2)$$ note: same roots!

If there is only one root z_1, then

$$az^2 + bz + c = a(z - z_1)^2$$

Examples:

1. $2z^2+z+1 = 2\left(z+\frac{1-\sqrt{7}i}{4}\right)\left(z+\frac{1+\sqrt{7}i}{4}\right)$
2. $z^2-5z+6 = (z-3)(z-2)$
3. $z^2+4z+4 = (z+2)^2$

That's why we call $z=-2$ a double root here: it appears <u>twice</u> in the factorization.

Solving polynomial equations

$P(z)=0$ where $P(z)$ is a polynomial of degree k:

$P(z) = a_k z^k + \cdots + a_1 z + a_0$

where $a_0, a_1, \ldots, a_k$ are complex numbers and $a_k \neq 0$.

Fundamental Theorem of Alg