§3. Complex numbers

In calculus, we typically work with **real numbers**.

Complex numbers are an extension of real numbers obtained by adding the **imaginary unit**

\[ i = \sqrt{-1} \]

(\( \sqrt{-1} \) does not exist in real numbers; the equation \( x^2 = -1 \) has no real solutions)

They turn out to have a lot of amazing properties, some of which we will use in this course.

§3.1. Basic algebraic properties

**Definition** A complex number is an expression of the form

\[ a + bi \]

where \( a, b \) are real numbers and \( i \) is the imaginary unit

**Examples:**
- Any real \( a \) is a complex number: \( a = a + 0 \cdot i \), e.g. \( 5 = 5 + 0 \cdot i \)
- \( i = 0 + 1 \cdot i \) is a complex number
For $z = a + bi$
we say that
$a$ is the real part of $z$
b is the imaginary part of $z$
and write $\text{Re } z = a$, $\text{Im } z = b$

Example: $\text{Re } 5 = 5$, $\text{Im } 5 = 0$
$\text{Re } i = 0$, $\text{Im } i = 1$

Graphical representation:
Complex number $a + bi \rightarrow$ point on the plane
with coordinates $(a, b)$

Algebraic operations

Addition/subtraction:
$(a + bi) + (c + di) = (a + c) + (b + d)i$
$(a + bi) - (c + di) = (a - c) + (b - d)i$
Example:
\[(1 + 3i) - (2 + i) = -1 + 2i\]

**Multiplication:** use that \(i^2 = -1\)
\[(a + bi)(c + di) = ac + bci + adi + bd i^2\]
\[= (ac - bd) + (bc + ad)i\]

**Examples:**
* \((2 + 3i)(3 - i) = 6 + 9i - 2i - 3i^2 = 9 + 7i\)
* \((1 + i)^2 = 2i\)

**Complex conjugate:** if \(z = a + bi\),
then the complex conjugate is \(\bar{z} = a - bi\)

**Example:** \(2 + 3i\) \(\bar{z} = 2 - 3i\)

**Absolute value:** if \(z = a + bi\)
then \(|z| = \sqrt{a^2 + b^2}\)

**Useful identity:** \[z \cdot \bar{z} = |z|^2\]

**Proof:** \((a + bi)(a - bi) = a^2 + b^2\)

**More identities:** \(\bar{z} + \bar{w} = \bar{z + w}, \bar{z} \cdot \bar{w} = \bar{z \cdot w}\)

\[|zw| = |z| \cdot |w|\]
Dividing complex numbers:

If \( z, w \) are complex, \( w \neq 0 \),
then compute \( \frac{z}{w} = \frac{z \cdot \bar{w}}{w \cdot \bar{w}} = \frac{z \cdot \bar{w}}{|w|^2} \)

\( \bar{w} \) is a real number, easy to divide.

Example: \( \frac{1 + 2i}{1 + i} = \frac{(1 + 2i)(1 - i)}{(1 + i)(1 - i)} \)

\[ = \frac{3 + i}{2} = \frac{3}{2} + \frac{1}{2} \cdot i. \]

Solving quadratic equations

\[ az^2 + bz + c = 0 \]

where \( a, b, c \) are real, \( a \neq 0 \)

Quadratic formula for the roots:

\[ z_{1,2} = \frac{-b \pm \sqrt{D}}{2a} \]

where \( D = b^2 - 4ac \)

3 cases:

\( D > 0 \): 2 real roots

\( D = 0 \): 1 root only, of multiplicity 2 (double root)

\( D < 0 \): 2 complex roots \( z_{1,2} = \frac{-b \pm \sqrt{-D} \cdot i}{2a} \)
Examples:

1. \(2z^2 + z + 1 = 0\)
   \(a = 2, \ b = 1, \ c = 1, \ D = b^2 - 4ac = -7 < 0\)
   The roots are
   \[z_1 = \frac{-1 + \sqrt{-7}i}{4}, \quad z_2 = \frac{-1 - \sqrt{-7}i}{4}\]
   Note: \(z_2 = \overline{z_1}\)

2. \(z^2 - 5z + 6 = 0\)
   \(a = 1, \ b = -5, \ c = 6, \ D = 1 > 0\)
   The roots are
   \[z_1 = \frac{5 + 1}{2} = 3, \quad z_2 = \frac{5 - 1}{2} = 2\]

3. \(z^2 + 4z + 4 = 0\)
   \(a = 1, \ b = 4, \ c = 4, \ D = 0\)
   Only one root: \(z = \frac{-4}{2} = -2\)

Factorization:

If \(z_1, z_2\) are the roots of the quadratic equation \(az^2 + bz + c = 0\) then
\[a(z - z_1)(z - z_2) \text{ or } a \text{ is the same roots!}\]
If there is only one root \(z_1\), then
\[a(z - z_1)^2\]
Examples:

1. \[ 2z^2 + z + 1 = 2\left(z + \frac{1 - \sqrt{7}i}{4}\right)\left(z + \frac{1 + \sqrt{7}i}{4}\right) \]

2. \[ z^2 - 5z + 6 = (z - 3)(z - 2) \]

3. \[ z^2 + 4z + 4 = (z + 2)^2 \]

That's why we call \( z = -2 \) a double root here: it appears twice in the factorization.

Solving polynomial equations:

P(z) = 0 where P(z) is a polynomial of degree k:

\[ P(z) = a_k z^k + \ldots + a_1 z + a_0 \]

where \( a_0, a_1, \ldots, a_k \) are complex numbers and \( a_k \neq 0 \).

Fundamental Theorem of Algebra:

Any polynomial equation as above has \( k \) complex solutions \( z_1, \ldots, z_k \) (not necessarily distinct) and

\[ P(z) = a_k \cdot (z - z_1)(z - z_2) \ldots (z - z_k) \]
Can we get a formula for the roots?

In general, no.

So we will mostly stick to quadratic equations.

But sometimes you can still find roots if you guess one or more roots by factorization:

**Example:** Solve $z^3 + 1 = 0$.

Guess one root $z_0 = -1$.

Then $z^3 + 1$ can be divided by $z + 1$:

$$z^3 + 1 = z^2(z + 1) - z^2 + 1 = z^2(z + 1) - z(z + 1) + z + 1 = (z^2 - z + 1)(z + 1).$$

So now we need to solve $z^2 - z + 1 = 0$ which is quadratic:

$$z_{1,2} = \frac{1 \pm \sqrt{1 - 4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

So the equation $z^3 + 1 = 0$ has roots $1, \frac{1 + \sqrt{3}i}{2}, \frac{1 - \sqrt{3}i}{2}$.

Factorization: $z^3 + 1 = (z + 1)(z - \frac{1 + \sqrt{3}i}{2})(z - \frac{1 - \sqrt{3}i}{2})$.