§3. Complex numbers

In calculus, we typically work with <u>real numbers</u>.

Complex numbers are an extension of real numbers obtained by adding the <u>imaginary unit</u>

$$i = \sqrt{-1}$$

($\sqrt{-1}$ does not exist in real numbers: the equation $x^2 = -1$ has no real solutions)

They turn out to have a lot of amazing properties, some of which we will use in this course.

§3.1. Basic algebraic properties

THEORY

<u>Definition</u> A complex number is an expression of the form

$a + bi$ where a, b are real numbers

and i is the imaginary unit

<u>Examples</u>:

- Any real a is a complex number: $a = a + 0 \cdot i$, e.g. $5 = 5 + 0 \cdot i$
- $i = 0 + 1 \cdot i$ is a complex number

For $z = a + bi$

we say that

a is the <u>real part</u> of z

b is the <u>imaginary part</u> of z

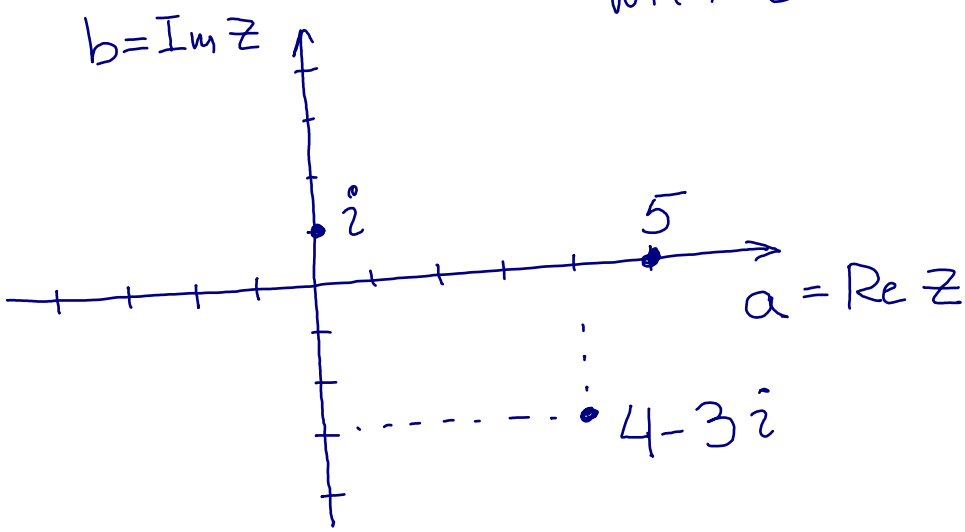
and write $\boxed{\operatorname{Re} z = a, \ \operatorname{Im} z = b}$

<u>Example:</u> $\operatorname{Re} 5 = 5, \ \operatorname{Im} 5 = 0$

$\operatorname{Re} i = 0, \ \operatorname{Im} i = 1$

Graphic representation:

Complex number $a+ib$ $\rightarrow$ point on the plane with coordinates (a,b)



Algebraic operations

Addition / subtraction:

$(a+bi) + (c+di) = (a+c) + (b+d)i$

$(a+bi) - (c+di) = (a-c) + (b-d)i$

Example:

$(1+3i)-(2+i) = -1+2i$

Multiplication: use that $i^2 = -1$

$(a+bi)(c+di) = ac + bci + adi + bdi^2$
$= (ac-bd) + (bc+ad)i$

Examples:

- $(2+3i)(3-i) = 6+9i-2i-3i^2$
  $= 9+7i$
- $(1+i)^2 = 2i$

Complex conjugate: if $z = a+bi$
then the complex conjugate is
$\bar{z} = a-bi$

Example: $\overline{2+3i} = 2-3i$

Absolute value: if $z = a+bi$
then $|z| = \sqrt{a^2+b^2}$

Useful identity: $\boxed{z \cdot \bar{z} = |z|^2}$

Proof: $(a+bi)(a-bi) = a^2+b^2$.

More identities: $\overline{z+w} = \bar{z}+\bar{w}$, $\overline{zw} = \bar{z}\,\bar{w}$

$|zw| = |z| \cdot |w|$

Dividing complex numbers:

if z, w are complex, $w \neq 0$

then compute $\dfrac{z}{w} = \dfrac{z \cdot \bar{w}}{w \cdot \bar{w}} = \dfrac{z \cdot \bar{w}}{|w|^2}$

multiply numerator & denominator by $\bar{w}$

$|w|^2$ is a real number, easy to divide

Example: $\dfrac{1+2i}{1+i} = \dfrac{(1+2i)(1-i)}{(1+i)(1-i)}$

$= \dfrac{3+i}{2} = \dfrac{3}{2} + \dfrac{1}{2} i.$

Solving quadratic equations

$az^2 + bz + c = 0$

where a, b, c are real, $a \neq 0$

Quadratic formula for the roots:

$z_{1,2} = \dfrac{-b \pm \sqrt{D}}{2a}$ where $D := b^2 - 4ac$

3 cases:

$D > 0$: 2 real roots

$D = 0$: 1 root only, of multiplicity 2 (double root)

$D < 0$: 2 complex roots $z_{1,2} = \dfrac{-b \pm \sqrt{-D} \cdot i}{2a}$

Examples:

1. $2z^2+z+1=0$

$a=2,\ b=1,\ c=1,\quad D=b^2-4ac=-7<0$

The roots are

$$z_1=\frac{-1+\sqrt{7}\,i}{4},\quad z_2=\frac{-1-\sqrt{7}\,i}{4}.$$

Note: $z_2=\bar{z}_1$

2. $z^2-5z+6=0$

$a=1,\ b=-5,\ c=6,\ D=1>0$

The roots are

$$z_1=\frac{5+1}{2}=3,\quad z_2=\frac{5-1}{2}=2$$

3. $z^2+4z+4=0$

$a=1,\ b=4,\ c=4,\ D=0$

Only one root: $z=\frac{-4}{2}=-2$

Factorization:

If z_1, z_2 are the roots of the quadratic equation $az^2+bz+c=0$ then

$$az^2+bz+c=a(z-z_1)(z-z_2)$$ ← note: same roots!

If there is only one root z_1, then

$$az^2+bz+c=a(z-z_1)^2$$

Examples:

1. $2z^2+z+1 = 2\left(z+\frac{1-\sqrt{7}i}{4}\right)\left(z+\frac{1+\sqrt{7}i}{4}\right)$
2. $z^2-5z+6 = (z-3)(z-2)$
3. $z^2+4z+4 = (z+2)^2$

that's why we call $z=-2$ a double root here: it appears <u>twice</u> in the factorization.

Solving polynomial equations

$P(z)=0$ where $P(z)$ is a polynomial of degree k:

$P(z) = a_k z^k + \cdots + a_1 z + a_0$

where $a_0, a_1, \ldots, a_k$ are complex numbers and $a_k \neq 0$.

$\boxed{\text{THEORY}}$

Fundamental Theorem of Algebra:

any polynomial equation as above has k complex solutions $z_1, \ldots, z_k$ (not necessarily distinct) and

$P(z) = a_k \cdot (z-z_1)(z-z_2) \cdots (z-z_k)$.

Can we get a formula for the roots?

In general, <u>no</u>.

So we will mostly stick to quadratic equations.

But sometimes you can still find roots if you guess one or more roots by factorization:

<u>Example:</u> Solve $z^3+1=0$.

Guess one root $z_0=-1$

Then z^3+1 can be divided by $z+1$:

$$\begin{aligned} z^3+1 &= z^2(z+1)-z^2+1 \\ &= z^2(z+1)-z(z+1)+z+1 \\ &= (z^2-z+1)(z+1). \end{aligned}$$

So now we need to solve $z^2-z+1=0$ which is quadratic: $z_{1,2}=\frac{1\pm\sqrt{1-4}}{2}=\frac{1\pm\sqrt{3}i}{2}$

So the equation $z^3+1=0$ has roots $1, \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}$

Factorization: $z^3+1=(z+1)\left(z-\frac{1+\sqrt{3}i}{2}\right)\left(z-\frac{1-\sqrt{3}i}{2}\right)$