§2.2. Linear first order ODEs

Standard form for a linear 1^{st} order ODE: $\boxed{\text{THEORY}}$

- Homogeneous case: $y' + a(t)y = 0$
- Inhomogeneous case: $y' + a(t)y = b(t)$

where $a(t)$, $b(t)$ are given functions

Some equations we saw before were linear but not in standard form.

e.g. $y' = 68 - y$ has the standard form

$y' + y = 68$ (i.e. $a(t) = 1$, $b(t) = 68$; inhomogeneous)

or $y' = y \cdot t^2$ has the standard form

$y' - t^2 \cdot y = 0$ (i.e. $a(t) = -t^2$, homogeneous)

Still, many equations are nonlinear,

e.g. $y' = y^2$

§2.2.1. Homogeneous equations

Those are separable (see §2.1):

$$y' + a(t)y = 0 \iff \frac{dy}{dt} = -a(t)y.$$

Following the algorithm for §2.1, we arrive to the following

Fact: the general solution to the equation $y' + a(t)y = 0$ is given by $\boxed{y(t) = Ce^{-A(t)}}$ where $A(t)$ is an antiderivative of $a(t)$ and C is an arbitrary constant

TECHNIQUE

ALGORITHM for finding the general solution to a linear homogeneous 1st order ODE:

Step 1: Write the ODE in the standard form

$$y' + a(t)y = 0$$

Step 2: find an antiderivative

$$A(t) = \int a(t)\,dt$$ (no need to add "+C")

Step 3: the general solution is

$$y(t) = Ce^{-A(t)}$$

Example: $y' = -ty$

Step 1: $y' + ty = 0 \Rightarrow a(t) = t$

Step 2: $A(t) = \int t\,dt = t^2/2$.

Step 3: $y(t) = Ce^{-t^2/2}$.

§2.2.2. Inhomogeneous equations

We solve these by variation of parameters.
I will first illustrate this on an example:

$$y' + y = 68$$

We first solve the corresponding homogeneous equation $y' + y = 0$:

$a(t) = 1 \Rightarrow A(t) = \int a(t)\,dt = \int dt = t$

General solution is Ce^{-t}, C = constant parameter

Now we look for solution of the inhomogeneous equation in the form

$y(t) = u(t)e^{-t}$ where u depends on t.

Plugging into the equation we get

THEORY + PRACTICE

$$y'(t) + y(t) = 68$$

$$\Downarrow$$

$$(u(t)e^{-t})' + u(t)e^{-t} = 68$$

$$\Downarrow$$

$$u'(t)e^{-t} - u(t)e^{-t} + u(t)e^{-t} = 68$$

Note the cancellation!

We get the equation which involves $u'(t)$ but not $u(t)$:

$$u'(t)e^{-t} = 68 \iff u'(t) = 68 \cdot e^{t}$$

Thus $u(t)$ is an antiderivative of $68 \cdot e^{t}$:

$$u(t) = \int 68e^{t}\,dt = 68e^{t} + C_0$$

C_0 is an arbitrary constant

Thus we get the general solution to the inhomogeneous equation:

$$y(t) = u(t)e^{-t} = (68e^{t} + C_0)e^{-t}$$
$$= 68 + C_0 e^{-t}.$$

ALGORITHM for finding the general solution to a linear inhomogeneous 1st order ODE

$$y' + a(t)y = b(t):$$

$\boxed{\text{TECHNIQUE}}$

Step 1: Write the general solution to the corresponding homogeneous ODE $y' + a(t)y = 0$

$$y(t) = Ce^{-A(t)}, \quad \text{where } A(t) = \int a(t)\,dt$$

Step 2: Look for solution to the inhomogeneous ODE in the form

$$(*)\ y(t) = u(t)e^{-A(t)}$$

replace the constant in the general homogeneous solution by a function of t

Step 3: Plug $(*)$ into the inhomogeneous ODE and get the $u(t)$ term to cancel, obtaining an equation on $u'(t)$:

$$y' + a(t)y = b(t)$$
$$(u(t)e^{-A(t)})' + a(t)u(t)e^{-A(t)} = b(t)$$
$$u'(t)e^{-A(t)} - \cancel{u(t)a(t)e^{-A(t)}} + \cancel{a(t)u(t)e^{-A(t)}} = b(t)$$

$$u'(t)e^{-A(t)} = b(t)$$

$$u'(t) = e^{A(t)}b(t)$$

Step 4: integrate to find $u(t)$
then plug into the formula (*)
to get $y(t)$:

$$u(t) = \int e^{A(t)}b(t) + C_0$$

$$y(t) = \left(\int e^{A(t)}b(t)\right)\cdot e^{-A(t)} + C_0 e^{-A(t)}$$

§2.2.3. An example: salting the tank

Problem A tank initially has 800L of fresh water (i.e. no salt). One pipe delivers a brine with concentration $75 g/L$ at a rate of $5 L/min$. Another pipe drains the tank at a rate of $3L/min$. The tank is constantly mixed, the concentration of salt is the same everywhere in the tank.

Find a formula for the concentration of salt in the tank after time t.

This problem has 2 parts:

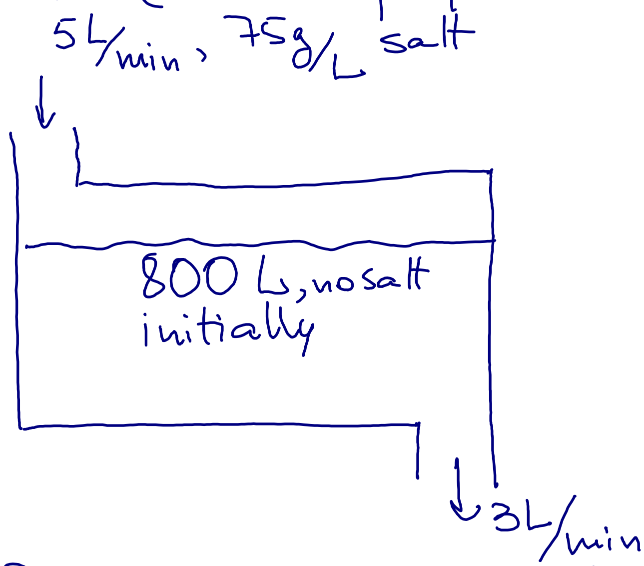
1. Modeling: get an ODE from the verbal description of the situation
2. Solving the ODE (in fact, an initial value problem)

TECHNIQUE PRACTICE

Steps for modeling

(this is <u>not</u> an algorithm; modeling is not straightforward)

<u>Step 1</u>: Draw a diagram of the system if possible (will help you with the later steps)



MITx: 1.1.10

<u>Step 2</u>: fix notation, specify units and unknowns

t — time, in minutes

$V(t)$ — volume of water in the tank, in Liters at time t

$y(t)$ — weight of salt in the tank, in grams at time t

Why weight and not concentration? will make the equation simpler...

$V(t)$ and $y(t)$ are the unknown functions

The concentration at time t is

$$\frac{y(t)}{V(t)}, \text{ in } g/L.$$

<u>Step 3</u>: Write the differential equation(s) for the unknown function(s)

Here we have 2 unknowns: $V(t)$ and $y(t)$

But $V(t)$ is easy to find:
the volume of water grows at the rate of $2\ L/min$ $(5-3=2)$ so

$$\boxed{V(t) = 800 + 2t}$$

For $y(t)$ we write an ODE:

$$\frac{dy}{dt} = \text{rate of change of weight of salt in the tank} = (\text{rate in}) - (\text{rate out})$$

Note: dy/dt has units of g/min

<u>Rate in</u> $= \underset{(L/min)}{5} \cdot \underset{(g/L)}{75} = \underset{(g/min)}{375}$

$$\text{Rate out} = \overset{(L/\text{min})}{3} \cdot \overset{(g/L)}{\frac{y(t)}{V(t)}}$$

3: rate out for volume of water

$\frac{y(t)}{V(t)}$: Concentration of salt in the tank

$$= \frac{3y(t)}{V(t)} = \frac{3y(t)}{800+2t}$$

We thus get the ODE

$$\boxed{\frac{dy}{dt} = 375 - \frac{3y}{800+2t}}$$

Since initially the tank had no salt, we also have the initial condition

$$\boxed{y(0) = 0}$$

Now we solve the ODE using the algorithm of §2.2.2:

MITx 1.2.7

PRACTICE

• Write in standard form:

$$y' + \frac{3}{800+2t}\, y = 375, \text{ i.e. } y' + a(t)y = b(t)$$

with $a(t) = \dfrac{3}{800+2t}$, $b(t) = 375$

18.03
§2.2
9

- Solve the homogeneous ODE:

$$A(t) = \int a(t)\,dt = \int \frac{3}{800+2t}\,dt$$

$$= \frac{3}{2}\int \frac{dt}{400+t} = \frac{3}{2}\ln(400+t)$$

General solution to the homogeneous ODE is

$$y(t) = Ce^{-A(t)} = C(400+t)^{-3/2}$$

- Solve the inhomogeneous ODE by variation of parameters:

Looking for solution in the form

$$y(t) = u(t)(400+t)^{-3/2}$$

Substituting into the equation, get

$$375 = y' + \frac{3}{800+2t}y = u'\cdot(400+t)^{-3/2} - \cancel{\frac{3}{2}u\cdot(400+t)^{-5/2}} + \cancel{\frac{3}{800+2t}u(400+t)^{-3/2}}$$

Get $375 = u'\cdot(400+t)^{-3/2}$

$$u' = 375\cdot(400+t)^{3/2}$$

Integrate to find u:

$$u(t) = 375\cdot\int(400+t)^{3/2}dt = 375\cdot\frac{2}{5}\cdot(400+t)^{5/2} + C_0$$

$$= 150\cdot(400+t)^{5/2} + C_0$$

Thus $y(t) = u(t)\cdot(400+t)^{-3/2}$

$= 150\cdot(400+t) + C_0\,(400+t)^{-3/2}$

18.03
§2.2
10

- Solve the initial value problem:

$0 = y(0) = 150\cdot 400 + C_0\cdot 400^{-3/2}$

So $C_0 = -\,150\cdot 400^{5/2} = -150\cdot 20^5$

$= -150\cdot 3200000 = -480000000$

So the solution is

$$\boxed{y(t) = 150\cdot(400+t) - 4.8\cdot 10^8\cdot(400+t)^{-3/2}}$$

And the concentration is (recall $V(t) = 800+2t$)

$$\boxed{\frac{y(t)}{V(t)} = 75 - 2.4\cdot 10^8\cdot(400+t)^{-5/2}}$$

Note that the concentration is an increasing function of $t \geq 0$ and as $t\to\infty$ it converges to 75 (g/L)