18.03
§ 2.1
1

GENERAL COMMENT:

Information in this course comes in 3 flavors:

- THEORY: explanation of why things are true and why we care about these. It is useful to know theory but you won't be tested too much on it (more advanced math courses are often almost 100% theory...) (18.032: version of 18.03 with emphasis on theory)
- TECHNIQUE: explanation of how to arrive to the solution to a problem. Knowing this is very important for success in this course
- PRACTICE: applying technique on examples, exercises etc. Practicing on your own/with study group/in recitations is also very important for success in the course.

§ 2. First order ODEs

THEORY

We are looking for a function $y = y(t)$ which solves an equation involving y and y'.

We write these in standard form

$$y'(t) = F(t, y(t))$$

where F is some given function

Other equations can be brought to standard form, for example

$$y \cdot t \cdot e^{y'} = 7 \longrightarrow \boxed{y' = \log\left(\frac{7}{y \cdot t}\right)} \leftarrow \text{standard form}$$

Not every ODE (even first order) can be solved by a formula but we will study some cases that can...

§2.1. Separable equations

Those are of the form

$$y' = f(y) \cdot g(t)$$

where f, g are given functions. That is,

$$y' = \left(\text{stuff depending only on } y\right) \cdot \left(\text{stuff depending only on } t\right)$$

derivative of y in t

Example 1: $\boxed{y' = y \cdot t^2}$

ALGORITHM for finding the general solution of a separable equation:

TECHNIQUE

General steps	Example
1. Rewrite the equation as $\frac{dy}{dt} = f(y) \cdot g(t)$	$\frac{dy}{dt} = y \cdot t^2$
2*. Move $f(y)$ to the left-hand side and dt to the right-hand side: $\frac{dy}{f(y)} = g(t)\,dt$ *can't divide by 0: see step 5	$\frac{dy}{y} = t^2 dt$
3. Integrate both sides. We get an arbitrary constant C: $\int \frac{dy}{f(y)} = \int g(t)\,dt + C$	$\int \frac{dy}{y} = \int t^2 dt + C$ $\ln\lvert y\rvert = \frac{t^3}{3} + C$
4. Solve the resulting (not differential anymore!) equation for y	$\lvert y\rvert = e^{t^3/3} \cdot e^C$ $y = \pm e^C \cdot e^{t^3/3}$ $y = \tilde{C} \cdot e^{t^3/3}$, $\tilde{C} \neq 0$ constant
5. In addition to Step 4 every number y_0 solving $f(y_0) = 0$ gives a stationary solution $y(t) = y_0$ (constant)	$y = 0$ is also a solution

In the above example, putting together steps 4 and 5 we get the general solution

$$\boxed{y(t) = \tilde{C} \cdot e^{t^3/3}}$$

where $\tilde{C}$ is an arbitrary constant

Now let us solve an <u>initial value problem (IVP)</u>:

<u>ALGORITHM for finding the solution to the initial value problem for a first order ODE (separable or not):</u>

TECHNIQUE

<u>1</u>. Find the <u>general solution</u>.
It will depend on some constant C.

<u>2</u>. Plug the initial condition to find C.

<u>3</u>. Put C into the general solution.

<u>Example:</u>

$$\begin{cases} y' = y \cdot t^2 & \leftarrow \text{ODE} \\ y(1) = 7 & \leftarrow \text{initial condition} \end{cases}$$

<u>1</u>. Found before:

$$y(t) = C \cdot e^{t^3/3}$$

<u>2</u>. $y(1) = 7 \Rightarrow C \cdot e^{1/3} = 7 \Rightarrow C = 7 \cdot e^{-1/3}$

<u>3</u>. $y(t) = 7 \cdot e^{-1/3} \cdot e^{t^3/3}$ is the solution to IVP

More examples of separable ODEs:

Example 2: $y' = 68 - y$ $\boxed{\text{PRACTICE}}$

Step 1: $\frac{dy}{dt} = 68 - y$

Step 2: $\frac{dy}{68-y} = dt$

Step 3: $\int \frac{dy}{68-y} = \int dt$

$-\ln|68-y| = t + C$

Step 4: Solving for y, get $y = 68 \pm e^{-(t+C)}$

Step 5: $y \equiv 68$ is also a solution

Steps 4+5 give the general solution

$$y(t) = 68 + \tilde{c}\, e^{-t}$$ (same as in §1...)

Example 3: $y' = y^2$

Step 1: $\frac{dy}{dt} = y^2$

Step 2: $\frac{dy}{y^2} = dt$

Step 3: $\int \frac{dy}{y^2} = \int dt + C$

$-\frac{1}{y} = t + C$

Step 4: solving for y, get $y = -\frac{1}{t+c} = \frac{1}{-c-t}$ (compare to §1)

Step 5: $y \equiv 0$ is also a solution