§11. Nonlinear ODEs: qualitative and numerical study

We now come back to systems of ODEs.

We study general non-linear systems

$\vec{y}' = \vec{F}(t, \vec{y})$ where

$\vec{y}(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$ is the unknown vector valued function

and $\vec{F}: \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a map.

Example: the system

$$\begin{cases} y_1' = y_1^2 + y_2 \\ y_2' = t y_1 \end{cases}$$

Can be written as $\vec{y}' = \vec{F}(t, \vec{y})$

where $\vec{F}(t, y_1, y_2) = \begin{pmatrix} y_1^2 + y_2 \\ t \cdot y_1 \end{pmatrix}$

§11.1. Autonomous systems

Here we study <u>autonomous systems</u>.
For these the right-hand side does not depend on t, i.e. they have the form

$\boxed{\text{THEORY}}$

$$\vec{y}' = \vec{F}(\vec{y})$$

for some map $F: \mathbb{R}^n \to \mathbb{R}^n$

§11.1.1. Phase portraits

When $n=2$ (i.e. $\vec{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$) we can draw the <u>phase portrait</u> of the system, which consists of all the trajectories of the system (similarly to the phase portraits of linear 2×2 systems we studied in §6.3)

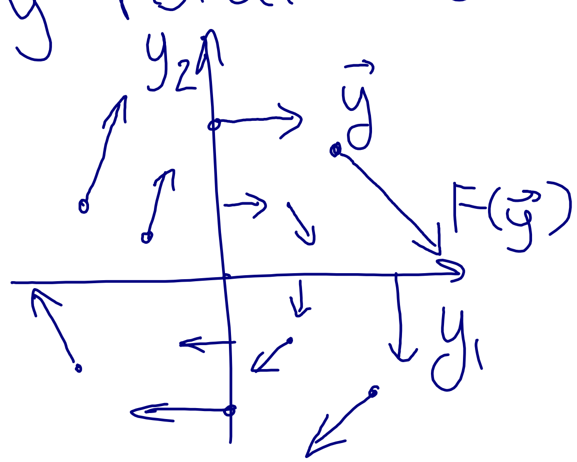
Note: since the lines in the phase portraits are the trajectories of the system $\vec{y}' = \vec{F}(\vec{y})$, at each point $\vec{y}$ the vector $\vec{F}(\vec{y})$ is tangent to the trajectory passing through $\vec{y}$.

This makes it possible to sketch the phase portrait by drawing the vectors $\vec{F}(\vec{y})$ with base points $\vec{y}$ at many different points $\vec{y}$.

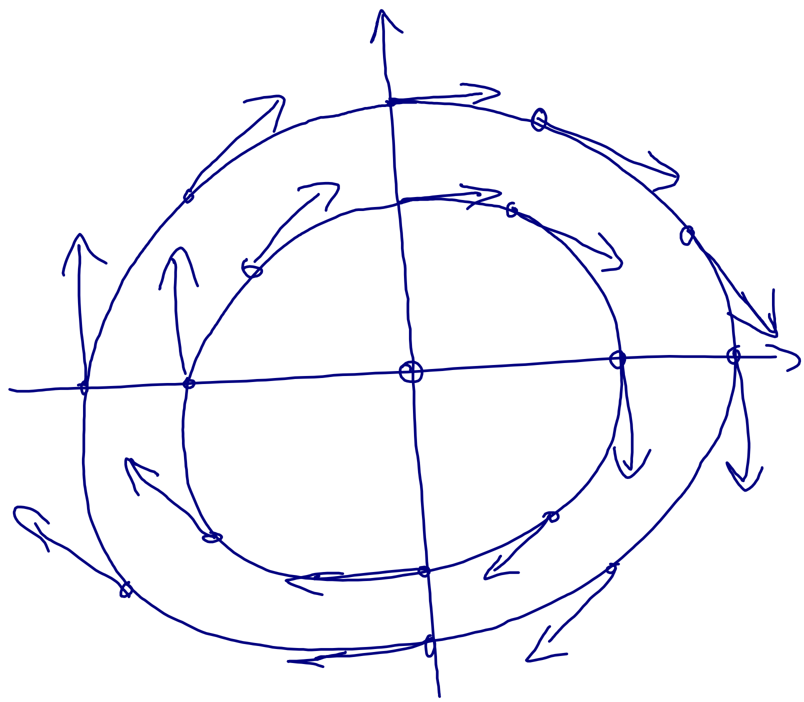
Example 1: $y_1' = y_2$, $y_2' = -y_1$ (harmonic oscillator)

$F(y_1, y_2) = (y_2, -y_1)$ That is,

$F(\vec{y}) = \vec{y}$ rotated clockwise by angle $\pi/2$.



This is compatible with the fact that trajectories are circles:



Example 2: $y_1' = y_1(1-y_1)$, $y_2' = y_1$

Here $\vec{F}(y_1, y_2) = (y_1(1-y_1), y_1)$.

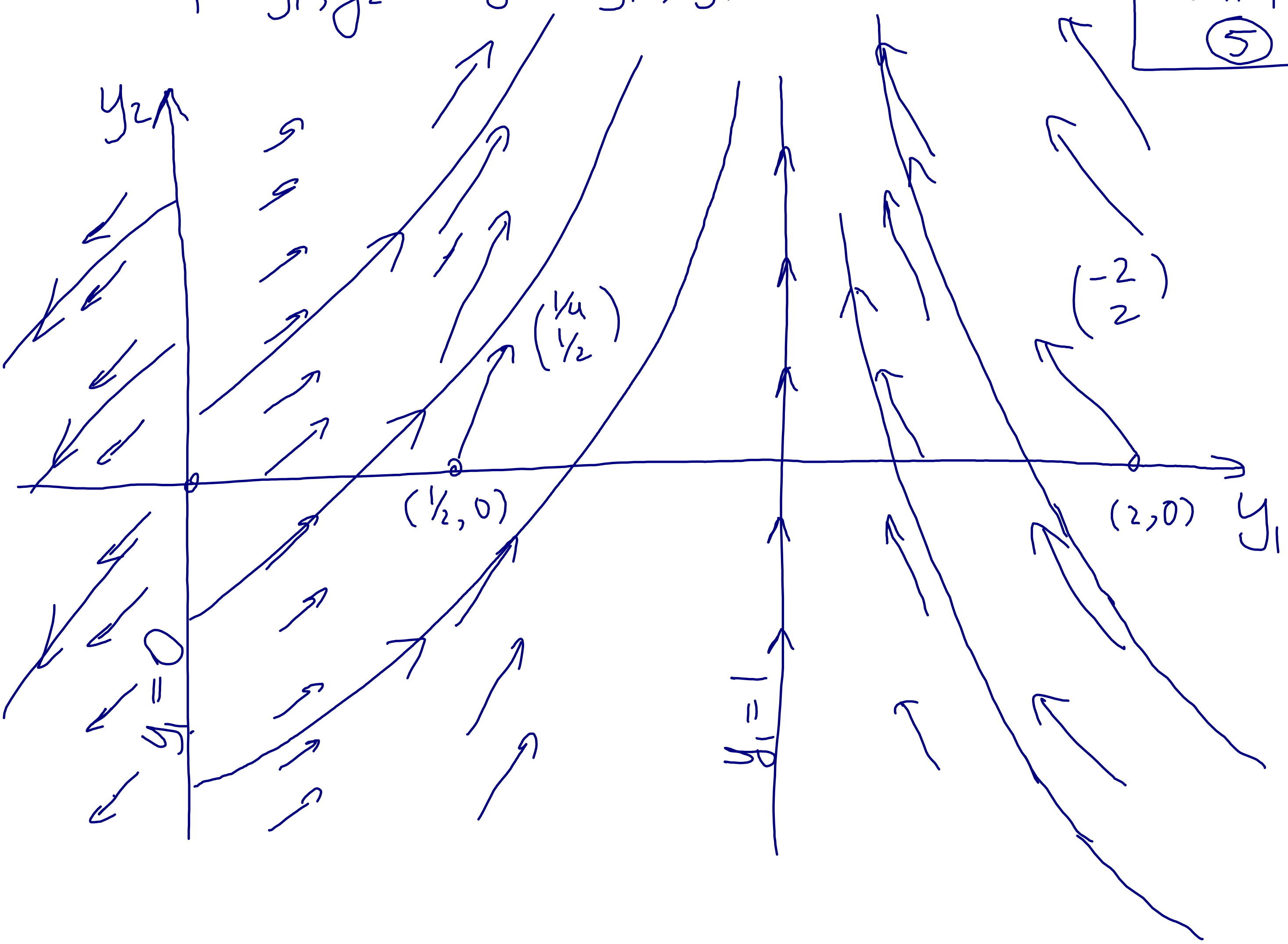
We have: $y_1 < 0 \Rightarrow y_1' < 0,\ y_2' < 0$

$0 < y_1 < 1 \Rightarrow y_1' > 0,\ y_2' > 0$

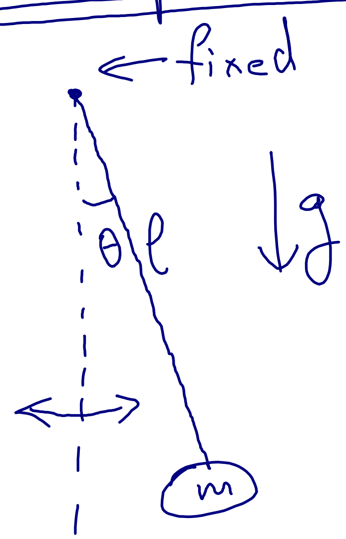
$y_1 > 1 \Rightarrow y_1' < 0,\ y_2' > 0$

We get the vectors $\vec{F}(\vec{y})$ looking like this:

$\vec{F}(y_1, y_2) = (y_1(1-y_1), y_1)$



§ 11.1.2. Example: the physical pendulum



$b \geq 0$ friction coefficient

ℓ = length of rod (in meters)

m = mass (in kg)

g = gravity of Earth (in m/sec^2)

t = time (in seconds)

$\theta(t)$ = angle at time t

To model the situation, we use <u>Newton's IInd law</u>:

m · acceleration = force

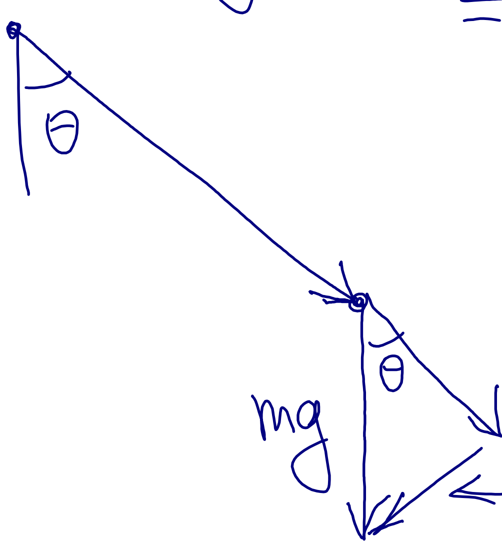
There is a force in the radial direction to keep the length constant:

So we get

$$m \cdot \ell \cdot \theta''(t) = m \cdot \text{tangential component of acceleration}$$
$$= \text{tangential component of the force}$$

There are 2 forces:

- <u>Gravity</u>: $\downarrow mg$. Tangential component = $= -mg \sin\theta(t)$



← tangential force $= -mg\sin\theta$

- Friction: we take it equal to

$$F = -b\,\ell\,\theta'$$

(the faster the pendulum is moving, the more the friction)

Putting these together we get

$$m\cdot\ell\cdot\theta'' = -m\cdot g\cdot\sin\theta - b\cdot\ell\cdot\theta',$$

i.e.

$$\boxed{\theta'' = -\frac{g}{\ell}\sin\theta - \frac{b}{m}\cdot\theta'}$$

This nonlinear 2nd order ODE is called the (damped) physical pendulum

We convert it to a 1st order system called the companion system:

use $\vec{y}(t) = \begin{pmatrix}\theta(t)\\ \omega(t)\end{pmatrix}$ where $\omega(t) = \theta'(t)$ (angular velocity)

We write

$$\begin{cases} \theta'(t) = \omega(t) \\ \omega'(t) = \theta''(t) = -\frac{g}{\ell}\sin\theta(t) - \frac{b}{m}\cdot\omega(t) \end{cases}$$

That is, $\vec{y} = \begin{pmatrix} \theta \\ \omega \end{pmatrix}$

solves the system $\vec{y}' = \vec{F}(y)$ where

$$\vec{F}(\theta, \omega) = \left(\omega, -\frac{b}{m}\omega - \frac{g}{\ell}\sin\theta\right)$$

i.e. $\begin{cases} \theta' = \omega \\ \omega' = -\frac{b}{m}\omega - \frac{g}{\ell}\sin\theta. \end{cases}$

Henceforth we assume that $m=1, \ell=1$,

So we get the system

$$\begin{cases} \theta' = \omega \\ \omega' = -b\omega - g\sin\theta \end{cases} \qquad (*)$$

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§11.1.3. Critical points, linearization, stability

For simplicity of notation we study here systems with only 2 unknown functions (The theory applies for >2 functions as well)

Definition A point $\vec{y}^{\circ}$ in $\mathbb{R}^2$ is called a critical point of the autonomous system $\vec{y}' = \vec{F}(\vec{y})$, if $\boxed{\vec{F}(\vec{y}^{\circ}) = 0}$

<u>Note:</u> if $\vec{y}^\circ$ is a critical pt. then $\vec{y}(t) = \vec{y}^\circ$ is a constant solution to $\vec{y}' = \vec{F}(\vec{y})$

<u>Example:</u> Critical points of the system (*) from <u>§11.1.2</u> are solutions to

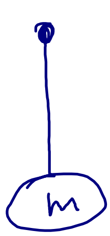
$$\begin{cases} \omega = 0 \\ -b\omega - g\sin\theta = 0. \end{cases}$$

That is, $\omega = 0$ and $\sin\theta = 0$.

Up to shifting θ by 2π (which does not change the position of the pendulum) we get 2 critical points:

$(0,0)$ and $(\pi, 0)$

$\theta = 0, \omega = 0$ $\theta = \pi, \omega = 0$



To understand the trajectories close to y^0, we introduce the linearization: THEORY

if $\vec{F}(y_1, y_2) = \begin{pmatrix} F_1(y_1, y_2) \\ F_2(y_1, y_2) \end{pmatrix}$

and $\vec{F}(\vec{y}^0) = 0$ where $\vec{y}^0 = \begin{pmatrix} y_1^0 \\ y_2^0 \end{pmatrix}$

then define the linearized system as $\vec{x}' = A\vec{x}$ where A is the 2×2 matrix

$$A = \begin{pmatrix} \partial_{y_1} F_1(y_1^0, y_2^0) & \partial_{y_2} F_1(y_1^0, y_2^0) \\ \partial_{y_1} F_2(y_1^0, y_2^0) & \partial_{y_2} F_2(y_1^0, y_2^0) \end{pmatrix}$$

Jacobi matrix

Note: if we write $\vec{y}(t) = \vec{y}^0 + \vec{x}(t)$ where $\vec{x}$ is small, then $\vec{x}$ solves

$$\vec{x}' = \vec{y}' = \vec{F}(\vec{y}) = \vec{F}(\vec{y}^0 + \vec{x}) = A\vec{x} + \cdots$$

where $(\cdots)$ denotes terms which are quadratic & higher in $\vec{x}$ (i.e. $|(\cdots)| \le C|\vec{x}|^2$)

Indeed,

$$F_1(y_1^0+x_1, y_2^0+x_2) = \underbrace{F_1(y_1^0, y_2^0)}_{0 \text{ since } \vec{y}^0 \text{ is a critical point}}$$

$$+ \underbrace{\partial_{y_1}F_1(y_1^0, y_2^0)\,x_1 + \partial_{y_2}F_1(y_1^0, y_2^0)\cdot x_2}_{= \text{first component of } A\cdot\vec{x}}$$

$$+ (\ldots) \leftarrow \text{bounded by } C|\vec{x}|^2$$

(This linear approximation formula is part of 18.02)
and a similar formula holds for F_2.
So the linearized system $\vec{x}' = A\vec{x}$ is obtained by throwing out the nonlinear terms.

The behavior of the nonlinear system $\vec{y}' = \vec{F}(\vec{y})$ near a critical point $\vec{y} = \vec{y}^0$ is qualitatively similar to the behavior of the linearized system $\vec{x}' = A\vec{x}$ near $\vec{x} = 0$. In particular:

Theorem Assume that $\vec{F}(\vec{y}^{\,o}) = 0$ and $\vec{x}' = A\vec{x}$ is the linearization of $\vec{y}' = F(\vec{y})$ at $\vec{y}^{\,o}$.
Assume that $\vec{x}' = A\vec{x}$ is stable.
Then $\vec{y}^{\,o}$ is a stable critical point of $\vec{y}' = F(\vec{y})$, i.e. for any solution $\vec{y}(t)$ with $\vec{y}(0)$ sufficiently close to $\vec{y}^{\,o}$, we have $\vec{y}(t) \longrightarrow \vec{y}^{\,o}$ as $t \to \infty$.

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§11.1.4. More on the physical pendulum

We come back to the system

$$(*)\begin{cases} \theta' = F_\theta(\theta,\omega) \overset{def}{=} \omega \\ \omega' = F_\omega(\theta,\omega) \overset{def}{=} -b\omega - g\sin\theta \end{cases}$$

$g>0$ gravity
$b\geq 0$ damping

Recall the 2 critical points $(0,0)$ & $(\pi,0)$

We study the linearizations at these points:

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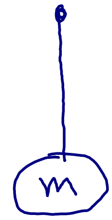
1. $\theta = 0,\ \omega = 0$
(mass at the bottom)



$F_\theta(\theta,\omega) = \omega \Rightarrow \partial_\theta F_\theta = 0,\ \partial_\omega F_\theta = 1$

$F_\omega(\theta,\omega) = -b\omega - g\sin\theta$

$\Rightarrow \partial_\theta F_\omega = -g\cos\theta,\ \partial_\omega F_\omega = -b$

Substituting $\theta = 0$, get $\partial_\theta F_\omega = -g$.

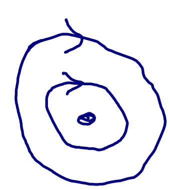
Thus the linearized system is $\vec{x}' = A\vec{x}$,

$$A = \begin{array}{c} \\ F_\theta \\ F_\omega \end{array} \begin{array}{c} \begin{array}{cc} \theta & \omega \end{array} \\ \begin{pmatrix} 0 & 1 \\ -g & -b \end{pmatrix} \end{array}$$

<u>Phase portrait of $\vec{x}' = A\vec{x}$?</u>

$\operatorname{tr} A = -b$, $\det A = g$

$b = 0 \Rightarrow$ get a <u>center</u>
(harmonic oscillator)



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$b > 0$ small $\Rightarrow$ get a <u>spiral sink</u>
(underdamped harmonic oscillator)

This critical point is stable when $b > 0$.

② $\Theta = \pi$, $\omega = 0$

The only change is in

$$\partial_\Theta F_\omega = -g\cos\Theta = g.$$

$$A = \begin{pmatrix} 0 & 1 \\ g & -b \end{pmatrix} \quad \operatorname{tr} A = -b \quad \det A = -g < 0.$$

The phase portrait of $\vec{x}' = A\vec{x}$ is a saddle, this critical point is unstable

Here is a phase portrait of the original nonlinear system for $b = 0$:

(See MITx 5.4.13 for the case $b > 0$)