§10.3. Theory of Fourier series

In §§10.1 - 10.2 we used the

$\boxed{\text{"}\Sigma\text{" means "sum"}}$

<u>Sine Fourier series</u>

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{\pi k}{L} x\right) = b_1 \sin\left(\frac{\pi}{L} x\right) + b_2 \sin\left(\frac{2\pi}{L} x\right) + \dots$$

$$0 \le x \le L$$

and the <u>cosine Fourier series</u>

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{\pi k}{L} x\right), \quad 0 \le x \le L$$

$\boxed{\text{THEORY}}$

Here $a_0, a_1, \dots$ and $b_1, b_2, \dots$ are some real constants.

We also used (but I did not explain) the formulas for the coefficients

$$b_k = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi k}{L} x\right) dx$$

$$a_k = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{\pi k}{L} x\right) dx$$

In this section we see where these formulas come from and study Fourier series in more detail.

Every function which has a continuous derivative on $[0,L]$ can be written as the sum of a sine Fourier series & it can also be written as the sum of a cosine Fourier series. The way in which the series converges is subtle, especially at the endpoints $0, L$, so we will not study it here.

§10.3.1. Finding the coefficients

Let's take a function f. We know that it can be written as a sine Fourier series

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{\pi k}{L} x\right), \quad 0 \le x \le L.$$

To get a formula for the coefficients b_k, we use

<u>Definition</u> Let f, g be real valued functions on $[0,L]$. Define their inner product

$$\langle f, g \rangle \stackrel{\text{def}}{=} \int_0^L f(x) g(x)\, dx.$$

The inner product on functions is analogous to the inner product on vectors in $\mathbb{R}^n$, defined by

$$\langle \vec{v}, \vec{w} \rangle \overset{\text{def}}{=\!=} \sum_{j=1}^{n} v_j w_j = v_1 w_1 + \cdots + v_n w_n, \quad \vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

In particular, we have the basic properties

- $\langle f, g \rangle = \langle g, f \rangle$
- $\langle f+g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$
- $\langle cf, g \rangle = c \langle f, g \rangle$ for any constant c

By analogy with $\mathbb{R}^n$, we say that two functions f, g are <u>orthogonal</u> on the interval $[0, L]$, if

$\langle f, g \rangle = 0$, i.e. if

$$\int_0^L f(x) g(x)\, dx = 0.$$

The following fact comes in handy in computing b_k:

Theorem: Consider the functions

$$v_k(x) = \sin\left(\frac{\pi k}{L} x\right), \quad 0 \le x \le L;\ k \ge 1 \text{ integer}$$

Then v_k form an orthogonal system:

$$\langle v_k, v_\ell \rangle = 0 \quad \text{for all } k \ne \ell.$$

Moreover, $\langle v_k, v_k \rangle = \frac{L}{2}$.

Proof (optional)

One way to establish orthogonality is by a direct computation of

$$\langle v_k, v_\ell \rangle = \int_0^L \sin\left(\frac{\pi k}{L} x\right) \sin\left(\frac{\pi \ell}{L} x\right) dx.$$

We however use a method involving integration by parts (IBP)

(The linear algebra part of this argument is similar to the proof that eigenvectors of a symmetric matrix form an orthogonal system)

Recall that v_k, v_ℓ are Dirichlet eigenfunctions of D^2 on $[0,L]$:

$$\begin{cases} v_k'' = \lambda_k v_k \\ v_k(0) = v_k(L) = 0 \end{cases} \qquad \begin{cases} v_\ell'' = \lambda_\ell v_\ell \\ v_\ell(0) = v_\ell(L) = 0 \end{cases}$$

$$\lambda_k = -\left(\frac{\pi k}{L}\right)^2 \neq \lambda_\ell = -\left(\frac{\pi \ell}{L}\right)^2.$$

Now we use IBP to show that

$$\langle v_k'', v_\ell \rangle = \langle v_k, v_\ell'' \rangle$$

Indeed, we have

$$\langle v_k'', v_\ell \rangle = \int_0^L v_k''(x)\, v_\ell(x)\, dx$$

$$= \int_0^L (v_k'(x))'\, v_\ell(x)\, dx \overset{\text{IBP}}{=} \underbrace{v_k'(x) v_\ell(x) \Big|_{x=0}^{L}}_{=0 \text{ as } v_\ell(0) = v_\ell(L) = 0}$$

$$- \int_0^L (v_k(x))' v_\ell'(x)\, dx \overset{\text{IBP}}{=} \underbrace{-v_k(x) v_\ell'(x) \Big|_{x=0}^{L}}_{=0 \text{ as } v_k(0) = v_k(L) = 0}$$

$$+ \int_0^L v_k(x)\, v_\ell''(x)\, dx = \langle v_k, v_\ell'' \rangle.$$

Now since $v_k'' = \lambda_k v_k$, $v_\ell'' = \lambda_\ell v_\ell$ we get

$$\lambda_k \langle v_k, v_\ell \rangle = \langle v_k'', v_\ell \rangle = \langle v_k, v_\ell'' \rangle = \lambda_\ell \langle v_k, v_\ell \rangle.$$

Since $\lambda_k \neq \lambda_\ell$, we get $\langle v_k, v_\ell \rangle = 0$.

Now to compute $\langle v_k, v_k \rangle$ we write

$$\langle v_k, v_k \rangle = \int_0^L \sin^2\left(\frac{\pi k}{L} x\right) dx = \text{Change of var's}$$

$s = \frac{\pi k}{L} x$

$dx = \frac{L}{\pi k} ds$

$$= \frac{L}{\pi k} \int_0^{\pi k} \sin^2 s \, ds$$

$$= \frac{L}{\pi k} \int_0^{\pi k} \frac{1 - \cos(2s)}{2} ds$$

$$= \frac{L}{\pi k} \cdot \left(\frac{s}{2} - \frac{\sin(2s)}{4} \right) \Bigg|_{s=0}^{\pi k} = \frac{L}{\pi k} \cdot \frac{\pi k}{2} = \frac{L}{2}.$$

A way to remember this formula is: the average value of $\sin^2\left(\frac{\pi k}{L} x\right)$ on the interval $[0, L]$ is equal to $\frac{1}{2}$. $\square$

Now we can compute b_ℓ in

$$f(x) = \sum_{\ell=1}^{\infty} b_\ell \sin\left(\frac{\pi \ell}{L} x\right).$$

Recall that $v_\ell(x) = \sin\left(\frac{\pi \ell}{L} x\right)$, so

$$f = \sum_{\ell=1}^{\infty} b_\ell v_\ell = b_1 v_1 + b_2 v_2 + b_3 v_3 + \cdots$$

Take the inner product with v_k:

$$\langle f, v_k\rangle = \left\langle \sum_{\ell=1}^{\infty} b_\ell v_\ell, v_k \right\rangle$$

$$= \sum_{\ell=1}^{\infty} b_\ell \langle v_\ell, v_k\rangle = b_k \langle v_k, v_k\rangle$$

Since $\langle v_\ell, v_k\rangle = 0$ when $\ell \neq k$.

Here we used properties of inner product:

$$\langle b_1 v_1 + b_2 v_2 + \cdots, v_k\rangle = b_1\langle v_1, v_k\rangle + b_2\langle v_2, v_k\rangle + \cdots$$

Recalling that $\langle v_k, v_k\rangle = \frac{L}{2}$ we arrive to

$$b_k = \frac{\langle f, v_k\rangle}{\langle v_k, v_k\rangle} = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{\pi k}{L}x\right)dx.$$

For <u>cosine</u> Fourier series

$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{\pi k}{L}x\right)$ we have

similar formulas $a_k = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{\pi k}{L}x\right)dx$

The factor $\frac{1}{2}$ in $\frac{a_0}{2}$ comes from the fact that

$$\int_0^L \cos^2\left(\frac{\pi k}{L}x\right)dx = \begin{cases} L, & \text{if } k=0 \\ L/2, & \text{if } k \geq 1 \end{cases}$$

§10.3.2 An example

We compute the sine and the cosine Fourier series on $[0,\pi]$ (i.e. $L=\pi$) for the function $\boxed{f(x)=1,\ 0\le x\le \pi}$

The <u>cosine Fourier series</u> is very simple:

$$f(x) = 1 + 0\cdot\cos x + 0\cdot\cos(2x) + \cdots$$

i.e. $a_0 = 2$, $a_k = 0$ for $k \geq 1$.

Indeed, we can compute this using

$$a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(kx)\,dx = \frac{2}{\pi}\int_0^{\pi}\cos(kx)\,dx.$$

The <u>sine Fourier series</u> is more complicated:

$$b_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,dx = \frac{2}{\pi}\int_0^{\pi}\sin(kx)\,dx$$

$$= \frac{2}{\pi}\cdot\left(-\frac{1}{k}\cos(kx)\right)\Big|_{x=0}^{\pi} = \frac{2}{\pi k}\left(\cos 0 - \cos(\pi k)\right)$$

$$= \frac{2}{\pi k}\left(1-(-1)^k\right) = \begin{cases} \frac{4}{\pi k}, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even} \end{cases}$$

That is, writing odd k as $k=2j+1$

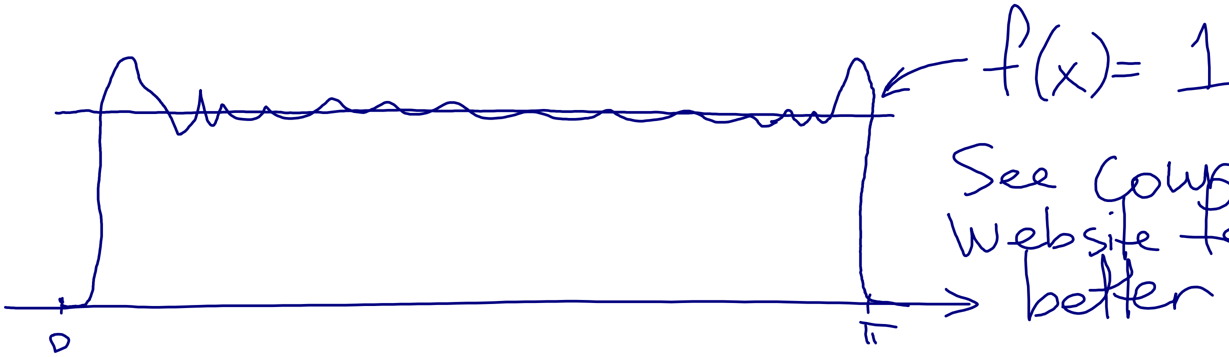
$$f(x) = \frac{4}{\pi}\sum_{j=0}^{\infty} \frac{\sin((2j+1)x)}{2j+1}, \quad 0 \le x \le \pi. \qquad (*)$$

Recall that $f(x)=1$ for all x in $[0,\pi]$

The convergence in $(*)$ is subtle:

- For $0<x<\pi$ the series converges conditionally to 1
- At the endpoints $x=0$, $x=\pi$ the sum of the series is equal to 0 (since each term is $=0$) but $f(x)=1$.

Here is a sketch of the partial sum of the series, $\frac{4}{\pi}\sum_{j=0}^{n}\frac{\sin((2j+1)x)}{2j+1}$, for large n:



See course website for a better plot

Here is an application:

Problem Solve the 1-BVP for the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, & t \geq 0, \quad 0 \leq x \leq \pi \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = 1 \end{cases}$$

these are incompatible at $t=0$, $x=0$ or $x=\pi$ but let's ignore this

Write $1 = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\sin((2j+1)x)}{2j+1}$

To get a solution of the heat equation, multiply j-th term by $e^{-(2j+1)^2 t}$:

$$u(x,t) = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{e^{-(2j+1)^2 t} \sin((2j+1)x)}{2j+1}.$$

Note: this models a rod of initial temperature 1 whose ends are suddenly exposed to temperature 0.

See the course website for a numerical illustration.

§ 10.3.3. Periodic Fourier series

For simplicity we now fix $L = \pi$.

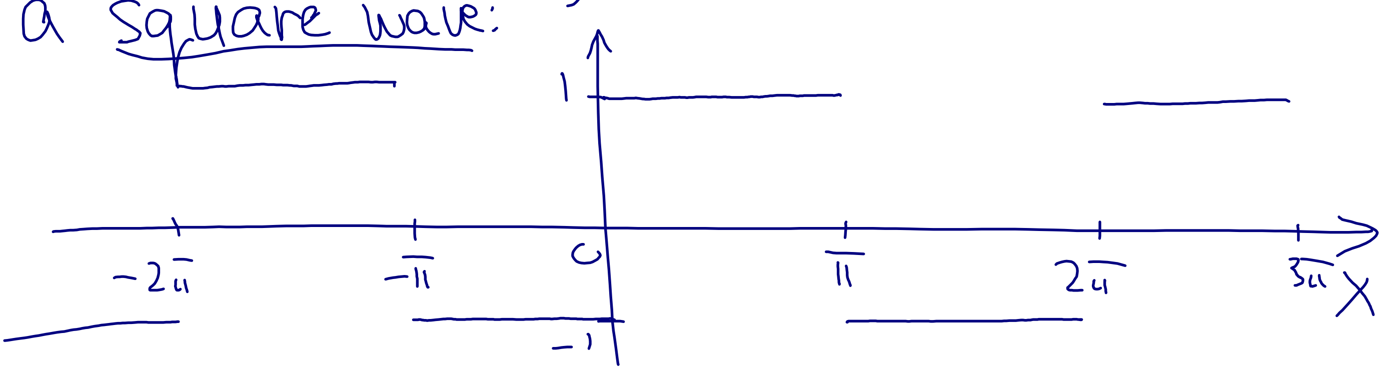
Sine Fourier series: $\displaystyle\sum_{k=1}^{\infty} b_k \sin(kx)$

Cosine Fourier series: $\displaystyle\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx)$

Both of these formulas actually make sense for all x, not just $0 \le x \le \pi$.
What kind of functions do they give?

- If $f(x) = \displaystyle\sum_{k=1}^{\infty} b_k \sin(kx)$, x in $\mathbb{R}$, then
  f is 2π-periodic: $f(x+2\pi) = f(x)$
  and f is odd: $f(-x) = -f(x)$

E.g. for the sine series for 1 on $[0, \pi]$,
$f(x) = \displaystyle\frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\sin((2j+1)x)}{2j+1}$, the function f is
a square wave:



- If on the other hand

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) \text{ then}$$

  - f is <u>2π-periodic</u> and
  - f is <u>even</u>: $f(-x) = f(x)$

Now, if we add the sine & cosine series together, we get the <u>periodic Fourier series</u>

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$$

which can be used to represent any differentiable 2π-periodic function f.

The coefficients are given by

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\,dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\,dx.$$