§10.3. Theory of Fourier series

In §§10.1 – 10.2 we used the

"Σ" means "sum"

<u>Sine Fourier series</u>

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{\pi k}{L} x\right) = b_1 \sin\left(\frac{\pi}{L} x\right) + b_2 \sin\left(\frac{2\pi}{L} x\right) + \dots$$

$0 \le x \le L$

and the <u>cosine Fourier series</u>

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{\pi k}{L} x\right), \quad 0 \le x \le L$$

Here $a_0, a_1, \dots$ and $b_1, b_2, \dots$ are some real constants.

$\boxed{\text{THEORY}}$

We also used (but I did not explain) the formulas for the coefficients

$$b_k = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi k}{L} x\right) dx$$

$$a_k = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{\pi k}{L} x\right) dx$$

In this section we see where these formulas come from and study Fourier series in more detail.

Every function which has a continuous derivative on $[0,L]$ can be written as the sum of a sine Fourier series & it can also be written as the sum of a cosine Fourier series. The way in which the series converges is subtle, especially at the endpoints $0, L$, so we will not study it here.

§10.3.1. Finding the coefficients

Let's take a function f. We know that it can be written as a sine Fourier series

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{\pi k}{L} x\right), \quad 0 \le x \le L.$$

To get a formula for the coefficients b_k, we use

<u>Definition</u> Let f, g be real valued functions on $[0,L]$. Define their <u>inner product</u>

$$\langle f, g \rangle \stackrel{\text{def}}{=} \int_0^L f(x) g(x)\,dx.$$

The inner product on functions is analogous to the inner product on vectors in $\mathbb{R}^n$, defined by

$$\langle \vec{v}, \vec{w} \rangle \overset{\text{def}}{=\!=} \sum_{j=1}^{n} v_j w_j = v_1 w_1 + \cdots + v_n w_n, \quad \vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

In particular, we have the basic properties

- $\langle f, g \rangle = \langle g, f \rangle$
- $\langle f+g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$
- $\langle cf, g \rangle = c \langle f, g \rangle$ for any constant c

By analogy with $\mathbb{R}^n$, we say that two functions f, g are <u>orthogonal</u> on the interval $[0, L]$, if

$$\langle f, g \rangle = 0, \text{ i.e. if}$$

$$\int_0^L f(x) g(x)\, dx = 0.$$

The following fact comes in handy in computing b_k:

Theorem: Consider the functions

$$v_k(x) = \sin\left(\frac{\pi k}{L} x\right), \quad 0 \le x \le L;\ k \ge 1 \text{ integer}$$

Then v_k form an orthogonal system:

$$\langle v_k, v_\ell \rangle = 0 \quad \text{for all } k \ne \ell.$$

Moreover, $\langle v_k, v_k \rangle = \frac{L}{2}$.

Proof (optional)

One way to establish orthogonality is by a direct computation of

$$\langle v_k, v_\ell \rangle = \int_0^L \sin\left(\frac{\pi k}{L} x\right) \sin\left(\frac{\pi \ell}{L} x\right) dx.$$

We however use a method involving integration by parts (IBP)

(The linear algebra part of this argument is similar to the proof that eigenvectors of a symmetric matrix form an orthogonal system)

Recall that v_k, v_ℓ are Dirichlet eigenfunctions of D^2 on $[0, L]$:

$$\begin{cases} v_k'' = \lambda_k v_k \\ v_k(0) = v_k(L) = 0 \end{cases} \qquad \begin{cases} v_\ell'' = \lambda_\ell v_\ell \\ v_\ell(0) = v_\ell(L) = 0 \end{cases}$$

$$\lambda_k = \left(\frac{\pi k}{L}\right)^2 \neq \lambda_\ell = \left(\frac{\pi \ell}{L}\right)^2.$$

Now we use IBP to show that

$$\langle v_k'', v_\ell \rangle = \langle v_k, v_\ell'' \rangle$$

Indeed, we have

$$\langle v_k'', v_\ell \rangle = \int_0^L v_k''(x)\, v_\ell(x)\, dx$$

$$= \int_0^L (v_k'(x))'\, v_\ell(x)\, dx \overset{\text{IBP}}{=} \underbrace{v_k'(x) v_\ell(x) \Big|_{x=0}^{L}}_{=0 \text{ as } v_\ell(0) = v_\ell(L) = 0}$$

$$- \int_0^L (v_k(x))' v_\ell'(x)\, dx \overset{\text{IBP}}{=} \underbrace{-v_k(x) v_\ell'(x) \Big|_{x=0}^{L}}_{=0 \text{ as } v_k(0) = v_k(L) = 0}$$

$$+ \int_0^L v_k(x) v_\ell''(x)\, dx = \langle v_k, v_\ell'' \rangle.$$

Now since $v_k'' = \lambda_k v_k$, $v_\ell'' = \lambda_\ell v_\ell$ we get

$$\lambda_k \langle v_k, v_\ell \rangle = \langle v_k'', v_\ell \rangle = \langle v_k, v_\ell'' \rangle = \lambda_\ell \langle v_k, v_\ell \rangle.$$

Since $\lambda_k \neq \lambda_\ell$, we get $\langle v_k, v_\ell \rangle = 0$.

Now to compute $\langle v_k, v_k \rangle$ we write

$$\langle v_k, v_k \rangle = \int_0^L \sin^2\left(\frac{\pi k}{L} x\right) dx = \quad \text{Change of var's} \quad s = \frac{\pi k}{L} x, \quad dx = \frac{L}{\pi k} ds$$

$$= \frac{L}{\pi k} \int_0^{\pi k} \sin^2 s \, ds$$

$$= \frac{L}{\pi k} \int_0^{\pi k} \frac{1 - \cos(2s)}{2} ds$$

$$= \frac{L}{\pi k} \cdot \left(\frac{s}{2} - \frac{\sin(2s)}{4} \right) \Bigg|_{s=0}^{\pi k} = \frac{L}{\pi k} \cdot \frac{\pi k}{2} = \frac{L}{2}.$$

A way to remember this formula is: the average value of $\sin^2\left(\frac{\pi k}{L} x\right)$ on the interval $[0, L]$ is equal to $\frac{1}{2}$. $\square$

Now we can compute b_ℓ in

$$f(x) = \sum_{\ell=1}^{\infty} b_\ell \sin\left(\frac{\pi \ell}{L} x\right).$$

Recall that $v_\ell(x) = \sin\left(\frac{\pi \ell}{L} x\right)$, so

$$f = \sum_{\ell=1}^{\infty} b_\ell v_\ell = b_1 v_1 + b_2 v_2 + b_3 v_3 + \cdots$$

Take the inner product with v_k:

$$\langle f, v_k \rangle = \left\langle \sum_{\ell=1}^{\infty} b_\ell v_\ell, \; v_k \right\rangle$$

$$= \sum_{\ell=1}^{\infty}$$

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