§10.2. The wave equation

The wave equation has the form

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

where $u(x,t)$ is the unknown function and $c>0$ is a given constant

The only difference from the heat equation is that $\frac{\partial u}{\partial t}$ is replaced by $\frac{\partial^2 u}{\partial t^2}$ but this changes completely the behavior of solutions.

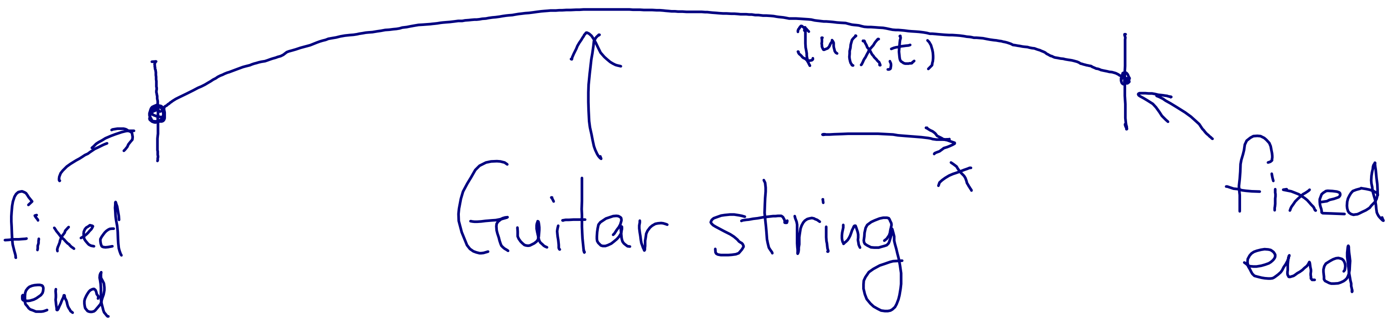
We will study the <u>initial-boundary value problem</u>

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t), \quad t \geq 0, \quad 0 \leq x \leq L \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = f(x) \\ \frac{\partial u}{\partial t}(x,0) = g(x) \end{cases}$$

$u(0,t)=0$, $u(L,t)=0$ → Dirichlet boundary conditions

$u(x,0)=f(x)$, $\frac{\partial u}{\partial t}(x,0)=g(x)$ → initial conditions, now there are <u>two</u> of them

§10.2.1. Modeling: a vibrating string



The shape of the string at time t is the graph of $u(x,t)$ as a function of x. (c = speed of wave propagation)

Roughly speaking, the wave equation models the behavior of the string because of Newton's IInd law:

$\frac{\partial^2 u}{\partial t^2}(x,t)$ = acceleration of a point on the string

$\frac{\partial^2 u}{\partial x^2}(x,t) \sim$ force acting on the string

See MITx for more details

Initial data: f = initial shape of the string

g = initial velocity

§10.2.2. The general solution

Now we find the general solution to

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}, \quad u = u(x,t), \ t \geq 0, \ 0 \leq x \leq L \\ u(0,t) = u(L,t) = 0 \end{cases}$$

As in the case of the heat equation we look for special solutions. Now these will have the form

$$u(x,t) = h(t) \sin\left(\frac{\pi k}{L} x\right).$$

Plugging into the wave equation, we get

$$h''(t)\sin\left(\frac{\pi k}{L}x\right) = \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} = -c^2\left(\frac{\pi k}{L}\right)^2 h(t)\sin\left(\frac{\pi k}{L}x\right)$$

Thus we need $h'' + \left(\dfrac{c\pi k}{L}\right)^2 h = 0$.

This is a harmonic oscillator with the general solution

$$h(t) = C_1 \cos\left(\frac{c\pi k}{L}t\right) + C_2 \sin\left(\frac{c\pi k}{L}t\right).$$

This gives the special solutions to the PDE:

$$u(x,t) = \left(C_1 \cos\left(\frac{c\pi k}{L}t\right) + C_2 \sin\left(\frac{c\pi k}{L}t\right)\right)\sin\left(\frac{\pi k}{L}x\right)$$

And the general solution to the PDE + boundary conditions is

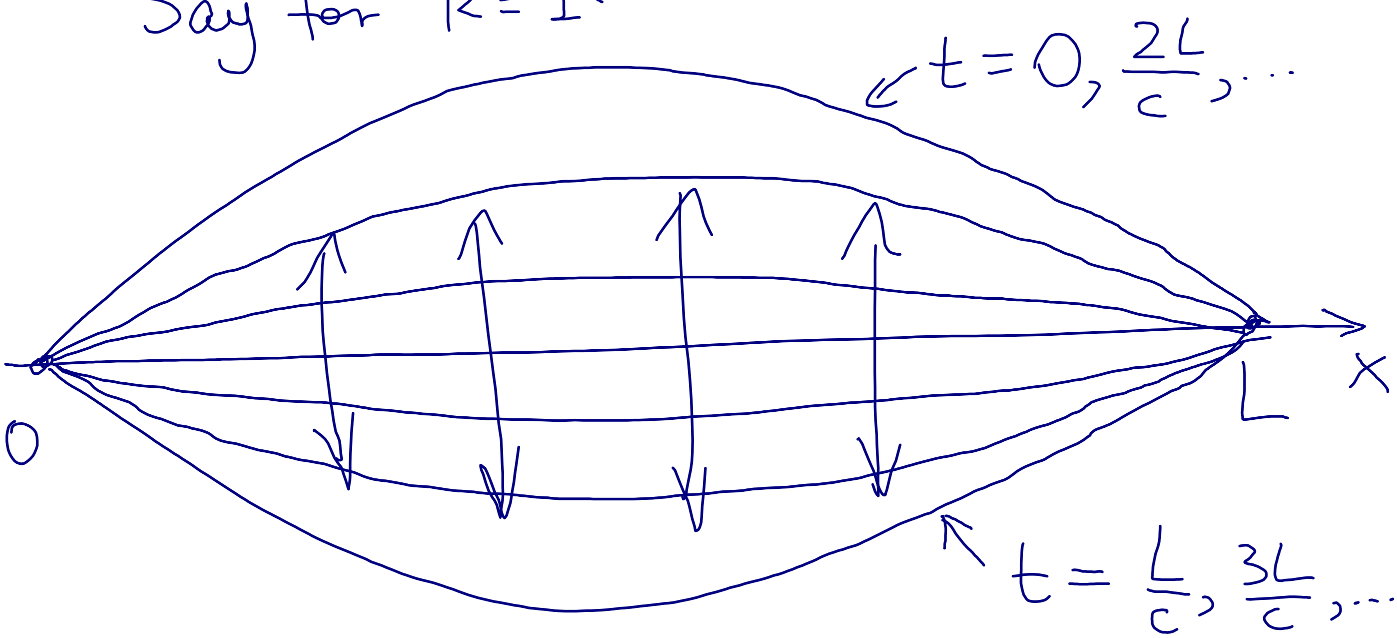
$$u(x,t) = \sum_{k=1}^{\infty}\left(b_k^{(1)}\cos\left(\frac{c\pi k}{L}t\right)+b_k^{(2)}\sin\left(\frac{c\pi k}{L}t\right)\right)\cdot\sin\left(\frac{\pi k}{L}x\right)$$

where $b_1^{(1)}, b_1^{(2)}, b_2^{(1)}, b_2^{(2)}, \ldots$ are some constants

§10.2.3. A musical interpretation

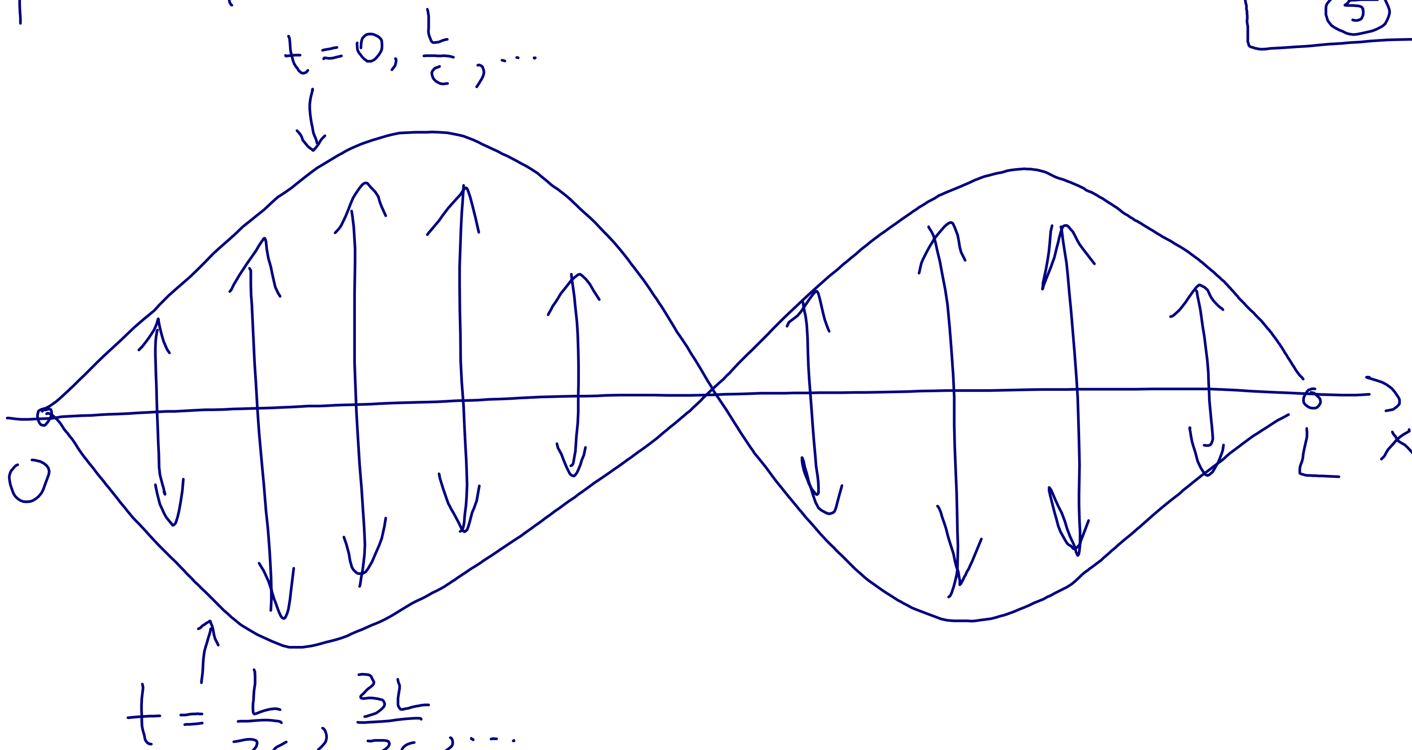
Let us look at just one special solution $\boxed{u(x,t) = \cos\left(\frac{c\pi k}{L}t\right)\sin\left(\frac{\pi k}{L}x\right)}$

Say for $k=1$:



$\updownarrow$ denotes the vibration of the string

For $k = 2$:



For each of these solutions the profile is the same with t but the amplitude is $\cos\left(\frac{c\pi k}{L} t\right)$, meaning that the string oscillates at the frequency $\omega_k = \frac{c\pi k}{L}$.

$k = 1$ is the base tone of the string

$k = 2, 3, \ldots$ are the overtones

ω_k = frequency (pitch) of the resulting sound.

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Analysing the formula

$\omega_k = \dfrac{c\pi k}{L}$ we see:

- Base tone $(k=1)$ gives the lowest pitch
- If we increase L (make the string longer) then the pitch gets lower
- We can also change the pitch by adjusting the tension of the string, which changes c.

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§10.2.4. Solving the I-BVP TECHNIQUE

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c\dfrac{\partial^2 u}{\partial x^2}, \quad u = u(x,t),\ t \geq 0,\ 0 \leq x \leq L \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) \end{cases}$$

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Recall the general solution:

$$u(x,t) = \sum_{k=1}^{\infty} \left(b_k^{(1)} \cos\left(\frac{c\pi k}{L} t\right) + b_k^{(2)} \sin\left(\frac{c\pi k}{L} t\right)\right) \sin\left(\frac{\pi k}{L} x\right)$$

To find $b_k^{(1)}$, $b_k^{(2)}$, plug in the initial conditions:

$$u(x,0) = \sum_{k=1}^{\infty} b_k^{(1)} \sin\left(\frac{\pi k}{L} x\right) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{k=1}^{\infty} b_k^{(2)} \cdot \frac{c\pi k}{L} \sin\left(\frac{\pi k}{L} x\right) = g(x).$$

Thus $b_k^{(1)}$, $\frac{c\pi k}{L} b_k^{(2)}$ are the coefficients of the sine Fourier series of f, g.

In particular

$$b_k^{(1)} = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi k}{L} x\right) dx$$

$$b_k^{(2)} = \frac{2}{c\pi k} \int_0^L g(x) \sin\left(\frac{\pi k}{L} x\right) dx$$