§10.2. The wave equation

The wave equation has the form

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]

where \( u(x,t) \) is the unknown function and \( c > 0 \) is a given constant. The only difference from the heat equation is that \( \frac{\partial u}{\partial t} \) is replaced by \( \frac{\partial^2 u}{\partial t^2} \) but this changes completely the behavior of solutions.

We will study the initial-boundary value problem

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t), \quad t > 0, \quad 0 \leq x \leq L \\
u(0,t) &= 0 \quad \text{Dirichlet boundary conditions} \\
u(L,t) &= 0 \\
u(x,0) &= f(x) \quad \text{initial conditions,}
\end{align*}
\]

\[ \frac{\partial u}{\partial t} (x,0) = g(x) \quad \text{two of them} \]
§10.2.1. Modeling: a vibrating string

The shape of the string at time $t$ is the graph of $u(x,t)$ as a function of $x$. ($c =$ speed of wave propagation)

Roughly speaking, the wave equation models the behavior of the string because of Newton's 2nd law:

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \text{acceleration of a point on the string}$$

$$\frac{\partial^2 u}{\partial x^2}(x,t) \sim \text{force acting on the string}$$

See MITx for more details

Initial data: $f =$ initial shape of the string
$g =$ initial velocity
§10.2.2. The general solution

Now we find the general solution to

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u=u(x,t), \quad t\geq 0, \quad 0 \leq x \leq L \]

\[ u(0,t)=u(L,t)=0 \]

As in the case of the heat equation, we look for special solutions. Now these will have the form

\[ u(x,t) = h(t) \sin\left(\frac{n \pi x}{L}\right). \]

Plugging into the wave equation, we get

\[ h''(t) \sin\left(\frac{n \pi x}{L}\right) = \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} = -c^2 \left(\frac{n \pi}{L}\right)^2 h(t) \sin\left(\frac{n \pi x}{L}\right) \]

Thus we need \[ h'' + \left(\frac{c^2 n \pi}{L}\right)^2 h = 0. \]

This is a harmonic oscillator with the general solution

\[ h(t) = C_1 \cos\left(\frac{c n \pi}{L} t\right) + C_2 \sin\left(\frac{c n \pi}{L} t\right). \]

This gives the special solutions to the PDE:

\[ u(x,t) = \left( C_1 \cos\left(\frac{c n \pi}{L} t\right) + C_2 \sin\left(\frac{c n \pi}{L} t\right) \right) \sin\left(\frac{n \pi x}{L}\right). \]
And the general solution to the PDE + boundary conditions is

\[ u(x,t) = \sum_{k=1}^{\infty} \left( b_k^{(1)} \cos\left(\frac{\pi k x}{L}\right) + b_k^{(2)} \sin\left(\frac{\pi k x}{L}\right) \right) \cdot \sin\left(\frac{\pi k}{L} t\right) \]

Where \( b_1^{(1)}, b_1^{(2)}, b_2^{(1)}, b_2^{(2)}, \ldots \)
are some constants

§ 10.2.3. A musical interpretation

Let us look at just one special solution

\[ u(x,t) = \cos\left(\frac{\pi k x}{L}\right) \sin\left(\frac{\pi k}{L} t\right) \]

Say for \( k = 1 \):

\[ t = 0, \frac{2L}{c}, \ldots \]

↑ denotes the vibration of the string
For $k = 2$:
\[ t = 0, \frac{L}{c}, \ldots \]
\[ t = \frac{L}{2c}, \frac{3L}{2c}, \ldots \]

For each of these solutions the profile is the same with $t$ but the amplitude is $\cos\left( \frac{c \pi k}{L} t \right)$, meaning that the string oscillates at the frequency $\omega_k = \frac{c \pi k}{L}$.

$k = 1$ is the base tone of the string $k = 2, 3, \ldots$ are the overtones $\omega_k =$ frequency (pitch) of the resulting sound.
Analysing the formula
\[ \omega_k = \frac{c k}{L} \]
we see:

- Base tone \((k=1)\) gives the lowest pitch.
- If we increase \(L\) (make the string longer), then the pitch gets lower.
- We can also change the pitch by adjusting the tension of the string, which changes \(c\).

§10.2.4. Solving the I-B VP Technique

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= c \frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t), \ t > 0, \ 0 \leq x \leq L \\
u(0,t) &= u(L,t) = 0 \\
u(x,0) &= f(x) \\
\frac{\partial u}{\partial t}(x,0) &= g(x)
\end{align*}
\]
Recall the general solution:

\[ u(x,t) = \sum_{k=1}^{\infty} \left( b_k^{(1)} \cos\left( \frac{c\pi k}{L} t \right) + b_k^{(2)} \sin\left( \frac{c\pi k}{L} t \right) \right) \sin\left( \frac{\pi k}{L} x \right) \]

To find \( b_k^{(1)} \), \( b_k^{(2)} \), plug in the initial conditions:

\[ u(x,0) = \sum_{k=1}^{\infty} b_k^{(1)} \sin\left( \frac{\pi k}{L} x \right) = f(x) \]

\[ \frac{\partial u}{\partial t}(x,0) = \sum_{k=1}^{\infty} b_k^{(2)} \cdot \frac{c\pi k}{L} \sin\left( \frac{\pi k}{L} x \right) = g(x) \]

Thus \( b_k^{(1)} \), \( \frac{c\pi k}{L} b_k^{(2)} \) are the coefficients of the sine Fourier series of \( f \), \( g \).

In particular

\[ b_k^{(1)} = \frac{2}{L} \int_{0}^{L} f(x) \sin\left( \frac{\pi k}{L} x \right) dx \]

\[ b_k^{(2)} = \frac{2}{c\pi k} \int_{0}^{L} g(x) \sin\left( \frac{\pi k}{L} x \right) dx \]