§10. PDEs & Fourier series

We now study partial differential equations (PDEs). The unknown function for a PDE is a function \( u \) of several variables and PDEs feature partial derivatives (from 18.02). PDEs are generally more difficult to study than ODEs, so we only study the following two linear equations in 1 time variable \( t \) and 1 space variable \( x \):

- The heat equation
  \[
  \frac{\partial u}{\partial t} = \Delta \frac{\partial u}{\partial x^2}
  \] (\( \Delta > 0 \) constant)

- The wave equation
  \[
  \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
  \] (\( c > 0 \) constant)
§10.1. The heat equation

§10.1.1. Setup

The heat equation is

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}
\]

where:

- \( t \geq 0 \) is the time variable
- \( x \) is the space variable.

We take \( x \) in an interval of length \( L \):

\( 0 \leq x \leq L \) where \( L > 0 \) is fixed

- \( u(x,t) \) is the unknown function defined for \( t \geq 0, 0 \leq x \leq L \).

Here is the picture of the domain of \( u \):

- \( J > 0 \) is a given constant
- \( \frac{\partial u}{\partial t} \) is the partial derivative of \( u \) in \( t \) (fix \( x \) & differentiate in \( t \))
- \( \frac{\partial^2 u}{\partial x^2} \) is the second partial derivative of \( u \) in \( x \) (fix \( t \) & differentiate in \( x \) twice)
10.1.2. Modeling

The heat equation models the temperature of a heated rod of length $L$:

Here $u(t, x) =$ temperature at time $t$ at position $x$ in the rod.

Units (sample):

- $t$ in seconds
- $x$ in meters
- $u(t, x)$ in °F
- $\frac{\partial u}{\partial t}$ in °F/sec
- $\frac{\partial^2 u}{\partial x^2}$ in °F/m²
- $\gamma$ in m²/sec is related to the heat conductivity of the rod.

Rough justification for the heat equation:

If $\frac{\partial^2 u}{\partial x^2} = 0$ then $\frac{\partial u}{\partial t} = 0$, i.e. $u(x, t)$ does not depend on $t$.

This corresponds to the linear distribution of temperature being a steady state:

$u \uparrow \quad \Rightarrow$ no change in temperature
For general $u(t, x)$, the second derivative $\frac{\partial^2 u}{\partial x^2}$ gives how far $u$ is from being linear in $x$, and the heat equation is pushing $u$ to become more linear as time goes on:

\[ \frac{\partial^2 u}{\partial x^2} < 0 \Rightarrow \frac{\partial u}{\partial t} < 0 \quad \frac{\partial^2 u}{\partial x^2} > 0 \Rightarrow \frac{\partial u}{\partial t} > 0 \]
(arrows indicate $\frac{\partial u}{\partial t}$)

§10.1.3. Initial-boundary value problem (1-BVP)

\[
\begin{cases}
\frac{\partial u}{\partial t} = \sqrt{\frac{\partial^2 u}{\partial x^2}}, & u = u(x,t), \ t \geq 0, \ 0 \leq x \leq L \\
u(x,0) = f(x) & \text{initial condition: fixing temperature at time 0} \\
u(0, t) = 0 & \text{Dirichlet boundary conditions: keeping the ends of the rod at temperature 0} \\
u(L, t) = 0 & \text{the rest of the rod is insulated}
\end{cases}
\]
Another option is to impose Neumann boundary conditions:
\[
\begin{align*}
\frac{\partial u}{\partial x}(0,t) &= 0 \\
\frac{\partial u}{\partial x}(L,t) &= 0
\end{align*}
\]
These correspond to insulating the entire rod.

Theorem For any (sufficiently regular) function \( f \) the I-BVP above has a unique solution \( u \).

Note: one actually does not have a solution for \( t < 0 \). This corresponds to the fact that thermodynamics is not time reversible (First law of thermodynamics).

§10.1.4. Special solutions

Recall that for a system of ODES \( \vec{y}' = A \vec{y} \) we found solutions in the form \( \vec{y}(t) = e^{\lambda t} \vec{v}(t) \) where \( A \vec{v} = \lambda \vec{v} \).

Now for the heat equation we look for solutions in the form
$u(x,t) = e^{\lambda t} v(x)$

Where $\lambda$ is a number and $v$ is a function. To satisfy the heat equation we need

$$x e^{\lambda t} v(x) = \frac{\partial u}{\partial t} = \nabla \cdot \nabla u = \nabla e^{\lambda t} \cdot v''(x)$$

That is,

$$v''(x) = \frac{1}{\lambda} v(x)$$

We also need $v$ to satisfy the boundary conditions. E.g. for Dirichlet b.c. we get

$$\begin{cases} v''(x) = \frac{1}{\lambda} v(x) , & 0 \leq x \leq L \\ v(0) = 0 \\ v(L) = 0 \end{cases}$$

That is, we want $\frac{1}{\lambda}$ to be an eigenvalue of $D^2$ on $[0,L]$ with the Dirichlet boundary conditions and $v$ to be the corresponding eigenfunction.

Recalling the computation in §9.2 we get

$$\frac{\lambda}{\lambda} = -\left( \frac{\pi k}{L} \right)^2$$

for some $k \geq 1$ integer

$$v(x) = C \sin \left( \frac{\pi k x}{L} \right).$$
That is, we can take
\[ \lambda = -\sqrt{\frac{n\pi}{L}}^2, \quad V(x) = \sin \left( \frac{n\pi}{L} x \right). \]
and
\[ u(x, t) = e^{-\sqrt{\frac{n\pi}{L}}^2 t} \sin \left( \frac{n\pi}{L} x \right) \]
Solves the heat equation
& the Dirichlet boundary conditions

8.10.1.5. The general solution
Recall that for \( \ddot{y} = Ay \) the general solution was
\[ \ddot{y}(t) = C_1 e^{\lambda_1 t} \sqrt{\lambda_1}, \ldots, C_n e^{\lambda_n t} \sqrt{\lambda_n}. \]
For the heat equation we have a similar "Theorem" The general solution to
\[ \begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad t \geq 0, \quad 0 \leq x \leq L \\
u(0, t) &= 0 \quad \text{could replace Dirichlet} \\
u(L, t) &= 0 \quad \text{with Neumann here but the formula below will change}
\end{align*} \]
has the form
\[ u(x, t) = \sum_{k=1}^{\infty} b_k e^{-\sqrt{\frac{n\pi}{L}}^2 t} \sin \left( \frac{n\pi}{L} x \right) \]
Where \( b_1, b_2, \ldots \) are constants.
(Why "Theorem"? Because we do not discuss conditions on \( b_k \) as \( k \to \infty \) which ensure convergence of the series.)
Remark: note that each
\[ e^{-\frac{1}{(\pi k)^2} t} \rightarrow 0 \quad t \rightarrow \infty \]

Thus \( u(t, x) \rightarrow 0 \) as \( t \rightarrow \infty \).

This corresponds to the fact that the rod will eventually reach the ambient temperature.

For Neumann b.c. we instead will have \( u(t, x) \rightarrow \text{const} \) as \( t \rightarrow \infty \).

\[ \S 10.1.6. \text{ Solving the I-BVP} \]

Here we come back to the initial-boundary value problem
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad u = u(x, t), \quad t \geq 0, \quad 0 \leq x \leq L \\
u(0, t) &= u(L, t) = 0 \\
u(x, 0) &= f(x).
\end{align*}
\]

The general solution is
\[
u(x, t) = \sum_{k=1}^{\infty} b_k e^{-\frac{1}{(\pi k)^2} t} \sin\left(\frac{\pi k}{L} x\right)
\]

To find \( b_k \), we use the initial condition, plugging in \( t = 0 \):
Definition. The Sine Fourier series on the interval \([0, L]\) has the form

\[
\sum_{k=1}^{\infty} b_k \sin \left( \frac{\pi k}{L} x \right).
\]

For \(b_k\) which decay sufficiently fast as \(k \to \infty\), this series converges and defines a function on the interval \([0, L]\).

We study Fourier series in more detail in §10.3 later. For now we just use the following:

Fact: if \(f\) is a (continuously differentiable) function on \([0, L]\) then \(f\) can be written as a sum of the sine Fourier series above with

\[
b_k = \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{\pi k}{L} x \right) dx.
\]

This leads to:
ALGORITHM for solving the I-BVP (x):

Step 1: Write the initial condition \( f \) as the sum of Fourier series:
\[
f(x) = \sum_{k=1}^{\infty} b_k \sin \left( \frac{\pi k}{L} x \right) \quad \text{where} \quad b_k = \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{\pi k}{L} x \right) dx
\]

Step 2: The solution \( u \) is given by
\[
u(x,t) = \sum_{k=1}^{\infty} b_k e^{-\left( \frac{\pi k}{L} \right)^2 t} \sin \left( \frac{\pi k}{L} x \right).
\]

Example: Solve the I-BVP
\[
\begin{align*}
\frac{\partial u}{\partial t} &= 2 \frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t), \quad t \geq 0, \quad 0 \leq x \leq \pi \\
u(0,t) &= u(\pi,t) = 0 \\
u(x,0) &= 2 \sin x - \sin(3x).
\end{align*}
\]

Solution: we have \( L = \pi, \quad \lambda = 2, \quad f(x) = 2 \sin x - \sin(3x) \)

Step 1: Write \( f(x) = \sum_{k=1}^{\infty} b_k \sin(kx) \)
where, rather than using the formula for \( b_k \), we just write
\[
b_1 = 2, \quad b_3 = -1,
\]
\[
b_k = 0 \quad \text{for all other} \quad k
\]
Step 2: to get $u(t,x)$ we start with the Fourier series for $f(x)$ and multiply each term $\sin(kx)$ by $e^{-2k^2t}$ to make sure it solves the heat equation:

$$f(x) = 2\sin x - \sin(3x)$$

$$u(x,t) = 2e^{-2t}\sin x - e^{-18t}\sin(3x).$$

This does solve the heat equation

$$\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2},$$

the boundary conditions

$$u(0,t) = u(L,t) = 0,$$

and the initial condition $u(x,0) = 2\sin x - \sin(3x)$

§ 10.1.7. Neumann boundary conditions

Now let us solve

$$\begin{cases}
\frac{\partial u}{\partial t} = \sqrt{\frac{\partial^2 u}{\partial x^2}}, & u = u(x,t), \ t \geq 0, \ 0 \leq x \leq L \\
\frac{\partial u}{\partial x}(0,t) = 0 \rightarrow \text{Neumann boundary conditions} \\
\frac{\partial u}{\partial x}(L,t) = 0 \\
u(x,0) = f(x)
\end{cases}$$
As in §10.1.4 we look for special solutions
\[ u(x,t) = e^{\lambda t} v(x) \] where
\[
\begin{aligned}
& v'' = \frac{\lambda}{D} v, \quad 0 \leq x \leq L \\
& v'(0) = 0 \\
& v'(L) = 0.
\end{aligned}
\]
That is, \( \lambda \) & \( v \) are eigenvalue & eigenfunction for \( D^2 \) on \([0,L]\) with Neumann boundary conditions.

Recalling the solution of this problem from §9.2, we get
\[
\lambda = -\pi \left( \frac{\pi k}{L} \right)^2, \quad k \geq 0 \text{ integer}
\]
\[ v(x) = \cos \left( \frac{\pi k}{L} x \right) \] (in particular, if \( k = 0 \), then \( v = 1 \))

This gives the general solution
\[ u(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k e^{-\left( \frac{\pi k}{L} \right)^2 t} \cos \left( \frac{\pi k}{L} x \right) \]
(we split off the \( k=0 \) term for later convenience)
where \( a_0, a_1, a_2, \ldots \) are some constants.
It remains to find the coefficients $a_k$. Plug in $t=0$:

$$U(x,0) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{\pi k}{L} x\right) = f(x).$$

This is called the cosine Fourier series. The coefficients are (see §10.3)

$$a_k = \frac{2}{L} \int_{0}^{L} f(x) \cos \left(\frac{\pi k}{L} x\right) \, dx.$$

In particular, $a_0 = \frac{2}{L} \int_{0}^{L} f(x) \, dx$. 