§10. PDEs & Fourier series

We now study partial differential equations (PDEs). The unknown function for a PDE is a function u of several variables and PDEs feature partial derivatives (from 18.02)

PDEs are generally more difficult to study than ODEs, so we only study the following two linear equations in 1 time variable t + 1 space variable x:

- The heat equation

$$\boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}}$$ ($\nu > 0$ constant)

- The wave equation

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$ ($c > 0$ constant)

§10.1. The heat equation

§10.1.1. Setup

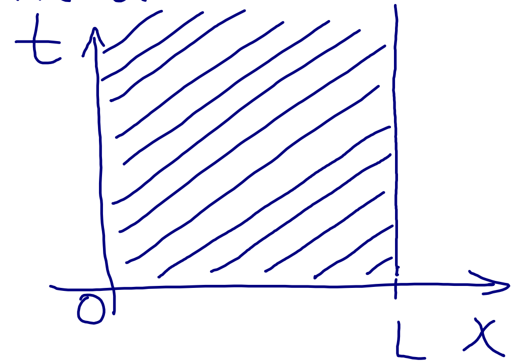
The heat equation is

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

THEORY

where:

- $t \geq 0$ is the time variable
- x is the space variable.
  We take x in an interval of length L:
  $0 \leq x \leq L$ where $L > 0$ is fixed
- $u(x,t)$ is the unknown function
  defined for $t \geq 0$, $0 \leq x \leq L$.
  Here is the picture of the domain of u:



- $\nu > 0$ is a given constant
- $\frac{\partial u}{\partial t}$ is the partial derivative of u in t
  (fix x & differentiate in t)
- $\frac{\partial^2 u}{\partial x^2}$ is the second partial derivative of u in x (fix t & differentiate in x twice)

§10.1.2. Modeling

The heat equation models temperature of a heated rod of length L:

Here $u(t,x)$ = temperature at time t at position x in the rod

Units (sample):

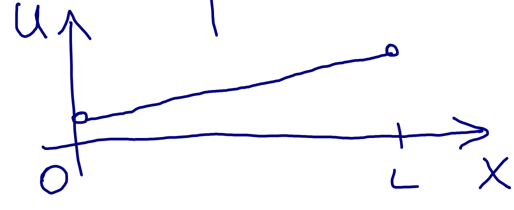
t in seconds	$\frac{\partial u}{\partial t}$ in $\frac{°F}{sec}$
x in meters	$\frac{\partial^2 u}{\partial x^2}$ in $\frac{°F}{m^2}$
$u(t,x)$ in °F	

ν in $\frac{m^2}{sec}$ is related to the heat conductivity of the rod

Rough justification for the heat equation:

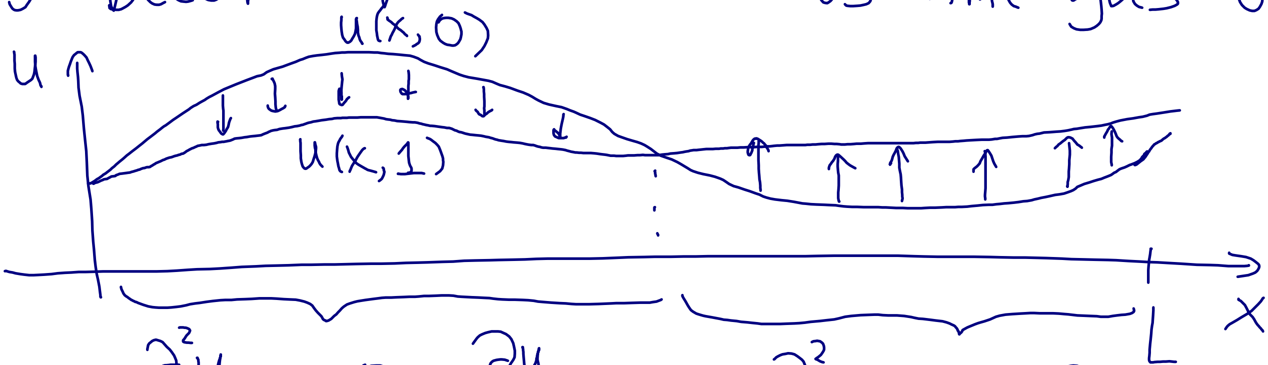
If $\frac{\partial^2 u}{\partial x^2} = 0$ then $\frac{\partial u}{\partial t} = 0$, i.e. $u(x,t)$ does not depend on t.

This corresponds to the linear distribution of temperature being a steady state:



$\Rightarrow$ no change in temperature

For general $u(t,x)$, the second derivative $\frac{\partial^2 u}{\partial x^2}$ gives how far u is from being linear in x, and the heat equation is pushing u to become more linear as time goes on:



$$\frac{\partial^2 u}{\partial x^2} < 0 \Rightarrow \frac{\partial u}{\partial t} < 0 \qquad \frac{\partial^2 u}{\partial x^2} > 0 \Rightarrow \frac{\partial u}{\partial t} > 0$$

(arrows indicate $\frac{\partial u}{\partial t}$)

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§10.1.3. Initial-boundary value problem (I-BVP)

$$\begin{cases} \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t),\ t \geq 0,\ 0 \leq x \leq L \\ u(x,0) = f(x) \\ u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$$

$u(x,0) = f(x)$ ← initial condition: fixing temperature at time 0

$u(0,t) = 0$, $u(L,t) = 0$ ← Dirichlet boundary conditions: keeping the ends of the rod at temperature 0 (the rest of the rod is insulated)

Another option is to impose
<u>Neumann boundary conditions</u>:

$$\begin{cases} \frac{\partial u}{\partial x}(0,t) = 0 \\ \frac{\partial u}{\partial x}(L,t) = 0 \end{cases}$$

These correspond to insulating the entire rod

<u>Theorem</u> For any (sufficiently regular) function f the I-BVP above has a unique solution u.

<u>Note</u>: one actually does not have a solution for $t<0$. This corresponds to the fact that thermodynamics is not time reversible (IInd law of thermodynamics)

<u>§10.1.4. Special solutions</u>

Recall that for a system of ODEs $\vec{y}' = A\vec{y}$ we found solutions in the form

$$\vec{y}(t) = e^{\lambda t}\,\vec{v}(t) \quad \text{where } A\vec{v} = \lambda\vec{v}$$

Now for the heat equation we look for solutions in the form

$$\boxed{u(x,t) = e^{\lambda t} v(x)}$$

where λ is a number and v is a function.

To satisfy the heat equation we need

$$\lambda e^{\lambda t} v(x) = \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} = \nu e^{\lambda t} \cdot v''(x)$$

That is, $\boxed{v''(x) = \frac{\lambda}{\nu} v(x)}$

We also need v to satisfy the boundary conditions. E.g. for Dirichlet b.c. we get

$$\begin{cases} v''(x) = \frac{\lambda}{\nu} v(x), & 0 \leq x \leq L \\ v(0) = 0 \\ v(L) = 0 \end{cases}$$

That is, we want $\frac{\lambda}{\nu}$ to be an eigenvalue of D^2 on $[0,L]$ with the Dirichlet boundary conditions and v to be the corresponding eigenfunction.

Recalling the computation in §9.2 we get $\boxed{\frac{\lambda}{\nu} = -\left(\frac{\pi k}{L}\right)^2}$ for some $k \geq 1$ integer

$$v(x) = c \sin\left(\frac{\pi k x}{L}\right).$$

That is, we can take

$$\lambda = -\nu\left(\frac{\pi k}{L}\right)^2, \quad v(x) = \sin\left(\frac{\pi k}{L}x\right).$$

and $\boxed{u(x,t) = e^{-\nu\left(\frac{\pi k}{L}\right)^2 t}\sin\left(\frac{\pi k}{L}x\right)}$

solves the heat equation & the Dirichlet boundary conditions

§10.1.5. The general solution

Recall that for $\vec{y}\,' = A\vec{y}$ the general solution was $\vec{y}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + \dots + C_n e^{\lambda_n t}\vec{v}_n$.

For the heat equation we have a similar

"Theorem" The general solution to

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial x^2}, \quad t \geq 0, \quad 0 \leq x \leq L \\ u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$$

$\leftarrow$ could replace Dirichlet with Neumann here but the formula below will change

has the form

$$\boxed{u(x,t) = \sum_{k=1}^{\infty} b_k e^{-\nu\left(\frac{\pi k}{L}\right)^2 t}\sin\left(\frac{\pi k}{L}x\right)}$$

where $b_1, b_2, \dots$ are constants.

(Why "Theorem"? Because we do not discuss conditions on b_k as $k \to \infty$ which ensure convergence of the series.)

Remark: note that each

$$e^{-\nu\left(\frac{\pi k}{L}\right)^2 t} \xrightarrow[t\to\infty]{} 0$$

Thus $u(t,x) \to 0$ as $t \to \infty$.
This corresponds to the fact that the rod will eventually reach the ambient temperature.
For Neumann b.c. we instead will have $u(t,x) \to$ const as $t \to \infty$.

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§10.1.6. Solving the I-BVP

Here we come back to the initial-boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial x^2}, \quad u = u(x,t), \ t \geq 0, \ 0 \leq x \leq L \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x). \end{cases} \qquad (*)$$

The general solution is

$$u(x,t) = \sum_{k=1}^{\infty} b_k e^{-\nu\left(\frac{\pi k}{L}\right)^2 t} \sin\left(\frac{\pi k}{L} x\right)$$

To find b_k, we use the initial condition, plugging in $t=0$:

$$u(x,0) = \boxed{\sum_{k=1}^{\infty} b_k \sin\left(\frac{\pi k}{L} x\right) = f(x)}$$

__Definition__. The $\boxed{\text{Sine Fourier series}}$ on the interval $[0, L]$ has the form

$$\boxed{\sum_{k=1}^{\infty} b_k \sin\left(\frac{\pi k}{L} x\right).}$$

For b_k which decay sufficiently fast as $k \to \infty$, this series converges and defines a function on the interval $[0, L]$.

We study Fourier series in more detail in §10.3 later. For now we just use the following

__Fact__: if f is a (continuously differentiable) function on $[0, L]$ then f can be written as a sum of the sine Fourier series above with $b_k = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi k}{L} x\right) dx.$

This leads to:

ALGORITHM for solving the I-BVP (*):

TECHNIQUE

Step 1: Write the initial condition f as the sum of Fourier series:

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{\pi k}{L} x\right) \text{ where } b_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{\pi k}{L}x\right)dx$$

Step 2: the solution u is given by

$$u(x,t) = \sum_{k=1}^{\infty} b_k e^{-\nu\left(\frac{\pi k}{L}\right)^2 t} \sin\left(\frac{\pi k}{L} x\right).$$

Example: Solve the I-BVP

$$\begin{cases} \dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}, & u = u(x,t),\ t \geq 0,\ 0 \leq x \leq \pi \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = 2\sin x - \sin(3x). \end{cases}$$

Solution: We have $L = \pi$, $\nu = 2$, $f(x) = 2\sin x - \sin(3x)$

Step 1: Write $f(x) = \sum_{k=1}^{\infty} b_k \sin(kx)$ where, rather than using the formula for b_k, we just write $b_1 = 2$, $b_3 = -1$,
$b_k = 0$ for all other k

Step 2: to get $u(t,x)$ we start with the Fourier series for $f(x)$ and multiply each term $\sin(kx)$ by e^{-2k^2t} to make sure it solves the heat equation:

$$f(x) = 2\sin x - \sin(3x)$$

$$u(x,t) = 2e^{-2t}\sin x - e^{-18t}\sin(3x).$$

This does solve the heat equation $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$, the boundary conditions $u(0,t) = u(\pi,t) = 0$, and the initial condition $u(x,0) = 2\sin x - \sin(3x)$

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§10.1.7. Neumann boundary conditions

Now let us solve

$$\begin{cases} \frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t),\ t \geq 0,\ 0 \leq x \leq L \\ \frac{\partial u}{\partial x}(0,t) = 0 \\ \frac{\partial u}{\partial x}(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

→ Neumann boundary conditions

As in §10.1.4 we look for special solutions

$$u(x,t) = e^{\lambda t} v(x) \quad \text{where}$$

$$\begin{cases} v'' = \frac{\lambda}{\nu} v, & 0 \le x \le L \\ v'(0) = 0 \\ v'(L) = 0. \end{cases}$$

That is, $\frac{\lambda}{\nu}$ & v are eigenvalue & eigenfunction for D^2 on $[0,L]$ with Neumann boundary conditions.

Recalling the solution of this problem from §9.2, we get

$$\lambda = -\nu \left(\frac{\pi k}{L}\right)^2, \quad k \ge 0 \text{ integer},$$

$$v(x) = \cos\left(\frac{\pi k}{L} x\right) \quad (\text{in particular, if } k=0, \text{ then } v = 1)$$

This gives the general solution

$$u(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k e^{-\nu \left(\frac{\pi k}{L}\right)^2 t} \cos\left(\frac{\pi k}{L} x\right)$$

(we split off the $k=0$ term for later convenience) where $a_0, a_1, a_2, \dots$ are some constants.

It remains to find the coefficients a_k. Plug in $t=0$:

$$u(x,0) = \underbrace{\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{\pi k}{L} x\right)}_{\text{This is called cosine Fourier series.}} = f(x).$$

The coefficients are (see §10.3)

$$a_k = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{\pi k}{L}x\right)dx.$$

In particular, $a_0 = \frac{2}{L}\int_0^L f(x)\,dx$.